Solving the dual-benchmark problem

Consider a pension fund. The fund seeks market returns but pays fixed-income liabilities. Given the choice between 2 portfolios having the same active risk and return but different absolute volatility, the fund would almost certainly (unless it is a hedge against other assets) be better off with the less volatile portfolio. Or consider the business of portfolio management. A fund is mandated to be around a benchmark, but it is performance against a peer group of competing funds that determines whether the fund attracts and retains assets. In both these cases, the standard mean-tracking variance objective does not fully reflect a portfolio manager's preferences. Adding a second benchmark, a risk-free asset in the first case, a composite of competitors in the second, captures what a manager wants more accurately. The literature presents a number of compelling arguments for dual benchmark risk control[1]; at Northfield, client requests have additionally convinced us of its practical importance.

The Framework

The starting point is the standard single-benchmark objective. A manager likes return and dislikes active risk and transaction costs.

$$\text{Utility}(p) = \alpha(p) - \lambda \text{TV}(p,b) - f(p)$$

where

p	= portfolio
b	= benchmark
α	= return
λ	= tracking risk aversion
TV(p,b)	= tracking variance between p & b
f(p)	= convex function for additional terms – transaction costs, etc.

Suppose a manager's preferences are known for 2 benchmarks considered in isolation from each another.

$$\text{Utility}_1(p) = \alpha(p) - \lambda_1 \text{TV}(p,b_1) - f(p)$$
$$\text{Utility}_2(p) = \alpha(p) - \lambda_2 \text{TV}(p,b_2) - f(p)$$

From the separate preferences, what is the joint utility?

1st Approach – Joint Risk Aversions

Assume the joint utility depends on the tracking variance penalties separately, i.e., that it is the single benchmark objective with an additional variance penalty and isolated risk aversions replaced by joint risk aversions.

$$\text{Utility}(p) = \alpha(p) - \hat{\lambda}_1 \text{TV}(p,b_1) - \hat{\lambda}_2 \text{TV}(p,b_2) - f(p)$$

Algebra reduces this to the standard single benchmark problem, with the new benchmark being the joint risk aversion weighted average of the benchmarks and the new risk aversion being the sum of the joint aversions. The only hitch is that the joint risk aversions aren't known.

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[1] see Richard Roll, "A Mean-Variance Analysis of Tracking Error", JPM, 1992.
Ming Yee Wang. "Multiple-Benchmark and Multiple-Portfolio Optimization", FAJ, Jan/Feb 1999.
George Chow. "Portfolio Selection Based on Return, Risk, and Relative Performance", FAJ, Mar/Apr 1995.

It turns out that naïve approaches, for example, setting the joint aversions to be the isolated aversions, result in portfolios that are too conservative or too aggressive. However, assuming that the joint aversions are proportional to the isolated aversions and setting their sum equal to the aversion weighted average of the isolated aversions leads to rational results. Since this formulation reduces to the standard single benchmark problem, it is easily solved by standard tools.

2nd Approach – Pareto Solutions

Instead of trading the importance of the separate objectives, look for portfolios that are preferable according to both criteria.

$$\text{Utility}(p) = \min[\text{Utility}_1(p), \text{Utility}_2(p)]$$

While the formulation is simple, at first look, there is no reason to believe that it is tractable. Fortunately, since the utility functions are concave and the feasible set is convex, the following key fact falls out:

> The maximum of the minimum of the functions is attained at either function's maximizer or where the functions are equal.

Finding either function's maximizer is the standard single benchmark problem. What about finding the maximizer subject to the constraint that the functions are equal? For the case where the risk aversions are the same, it is a linear constraint (easily solvable). When they are different, the constraint is the surface of an ellipsoid (hard). We are presently investigating another method of solving the optimization problem, relying on the fact that the minimum of concave functions is concave.

Maximum Tracking Error Against Multiple Benchmarks

A last topic is to maximize alpha less costs subject to tracking error constraints against multiple benchmarks. Each tracking error constraint describes a convex set. The intersection of convex sets is convex. Since a concave function has no local maxima over a convex set, the problem is solvable by nonlinear optimizers if not portfolio optimizers.

While both the literature and practioners agree that tracking variance against a single benchmark does not fully capture portfolio managers' investment objectives, expressing these beliefs is not trivial and can be ambiguous. There are a number of possible formulations; fortunately, these are often solvable by standard tools. Parker Shectman and I are finalizing a more detailed and technical paper on the topic. If you are interested, feel free to contact either one of us.

The Dual Benchmark Problem

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Overview of Talk

- Cases where multiple benchmarks occur
- Ways to express utility
- How to solve the optimization problems

Literature Review

- Ming Yee Wang. “Multiple-Benchmark and Multiple-Portfolio Optimization”, *FAJ*, Jan/Feb 1999.
- George Chow. “Portfolio Selection Based on Return, Risk, and Relative Performance” *FAJ*, Mar/Apr 1995.
- Richard Roll, “A Mean-Variance Analysis of Tracking Error”, *JPM*, 1992.

Cases for Multiple Benchmarks

- Mutual fund relative performance:
  - Managing both absolute and relative risk.
  - Managing index-relative and peer-relative risk.
  - Pursue a ‘model portfolio’ of fundamentally sound stock picks, while controlling risk against a benchmark index.
- University endowment funds:
  - Pay the bills, but also
  - create wealth for future expansion of the school.
- Pension fund:
  - Maximize likelihood of meeting mention obligations, while
  - minimizing sponsor’s costs.

Standard 1 Benchmark Framework

$$\text{Utility}(p) = \alpha(p) - \lambda TV(p,b) - f(p)$$

p = portfolio

b = benchmark

α = return

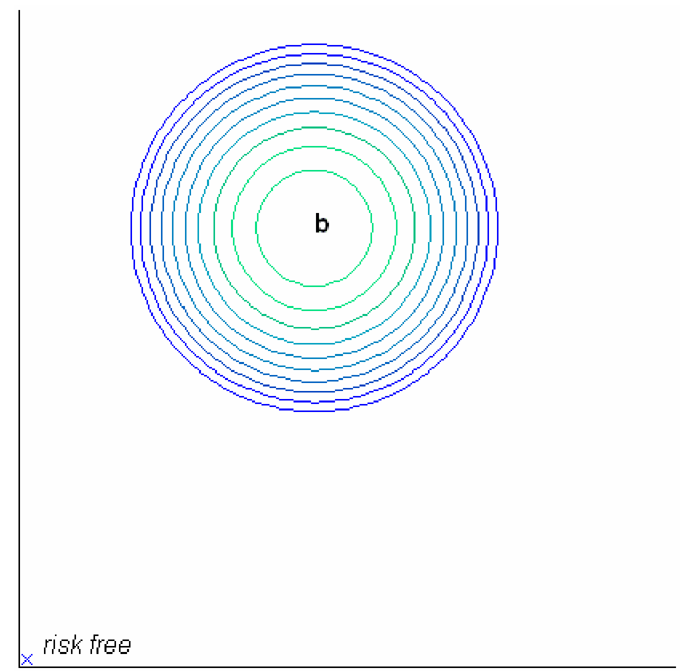
λ = tracking risk aversion

$TV(p,b)$ = tracking variance between p & b

$f(p)$ = convex function for additional terms - tcosts…

1 Benchmark Variance Penalty

- plot (mean subtracted) returns as a point – each period corresponds to different axis
- the distance between 2 points is the tracking error * sqrt(# periods)
- each ring out from the benchmark return requires 1 unit of alpha
- bigger $\lambda \Rightarrow$ rings are tighter



Extending to 2 Benchmarks

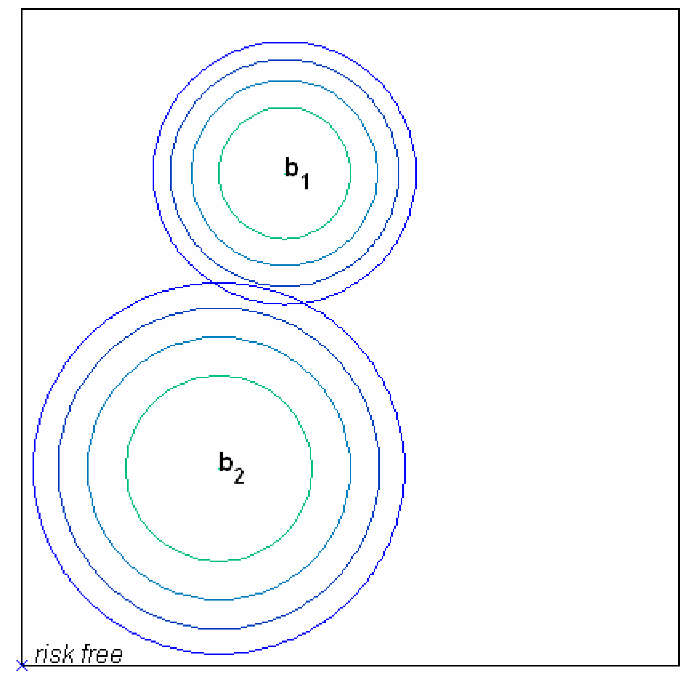
What do we like?

(Not clear even if we know our utility against each of the benchmarks taken separately)

$U_1(p) = \alpha(p) - \lambda_1 TV(p,b_1) - f(p)$

$U_2(p) = \alpha(p) - \lambda_2 TV(p,b_2) - f(p)$

=> $U(p) = \alpha(p) - ??? - f(p)$



1st Approach – Joint Risk Aversions

Suppose we still want to penalize tracking variances.

$$U(p) = \alpha(p) - \lambda'_1 TV(p,b_1) - \lambda'_2 TV(p,b_2) - f(p)$$

Good news – this becomes the single benchmark problem

$$U(p) = \alpha(p) - \lambda TV(p,b) - f(p)$$

where $\lambda = \lambda'_1 + \lambda'_2 \quad b = \lambda'_1/\lambda\, b_1 + \lambda'_2/\lambda\, b_2$

But how to determine joint λ's?

Settings That Don't Work

- Applying the original risk aversions will yield portfolios that are too conservative.

eg. b_1 is close to b_2, $\lambda_1 = \lambda_2 = \lambda$

$$U(p) = \alpha(p) - \lambda TV(p,b_1) - \lambda TV(p,b_2) - f(p)$$
$$\approx \alpha(p) - 2\lambda TV(p,b_1) - f(p)$$

- Averaging the risk aversions (same as adding U_1 & U_2) can yield too risky portfolios.

eg. b_1 is close to b_2, $\lambda_1 \gg \lambda_2$

$$U(p) = \alpha(p) - \lambda_1/2\ TV(p,b_1) - \lambda_2/2\ TV(p,b_2) - f(p)$$
$$\approx \alpha(p) - \lambda_1/2\ TV(p,b_1) - f(p)$$

A Rational Scaling

$U(p) = \alpha(p) - \lambda TV(p,b) - f(p)$

Assume joint λ's are proportional to separate λ's.

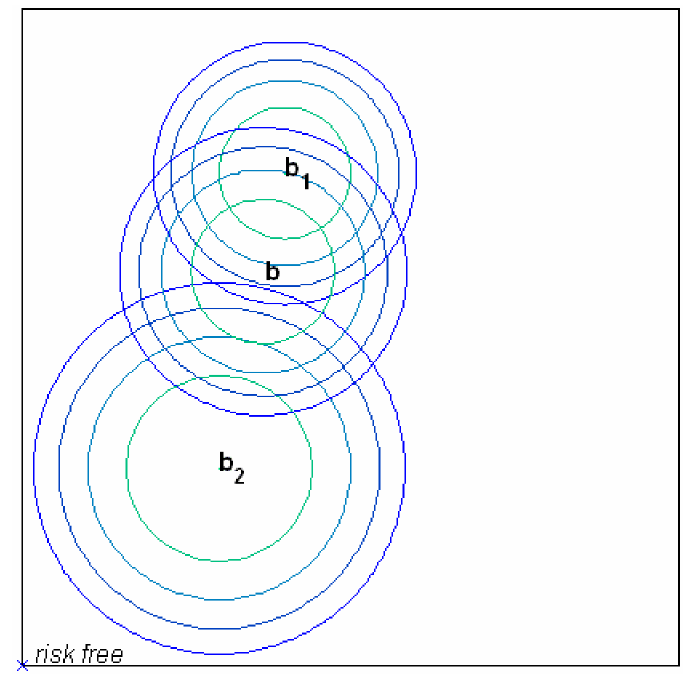
$$b = \lambda_1/(\lambda_1 + \lambda_2)\, b_1 + \lambda_2/(\lambda_1 + \lambda_2)\, b_2$$

set λ to the weighted average of the separate λ's.

$$\lambda = \lambda_1/(\lambda_1 + \lambda_2)\, \lambda_1 + \lambda_2/(\lambda_1 + \lambda_2)\, \lambda_2$$

$\Rightarrow$ Limiting cases yield correct answers and level sets look reasonable

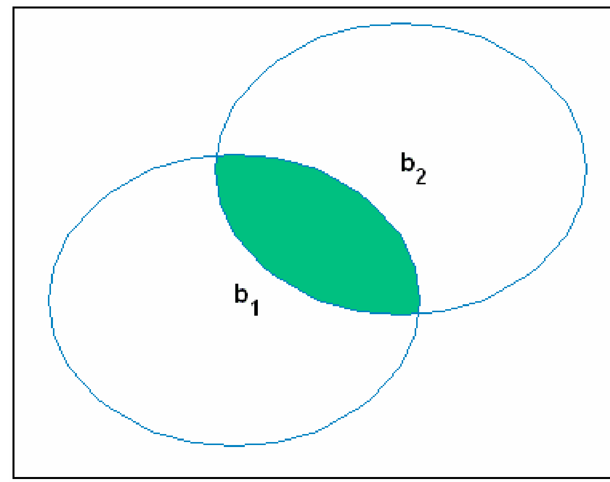
$\Rightarrow$ *Solved by a portfolio optimizer*



Maximize Return Subject To Tracking Error Constraints

Suppose we want to maximize return subject to tracking error constraints.

$$\begin{aligned} \text{maximize} \quad & U(p) = \alpha(p) - f(p) \\ \text{subject to} \quad & TV(p,b_1) \leq M_1 \\ & TV(p,b_2) \leq M_2 \end{aligned}$$



The constraints describe a convex set, so the problem is still maximizing a concave function over a convex set. *Solvable by general optimizers, but not portfolio optimizers.*

2nd Approach – Pareto Solutions

Given utility functions for each benchmark taken separately

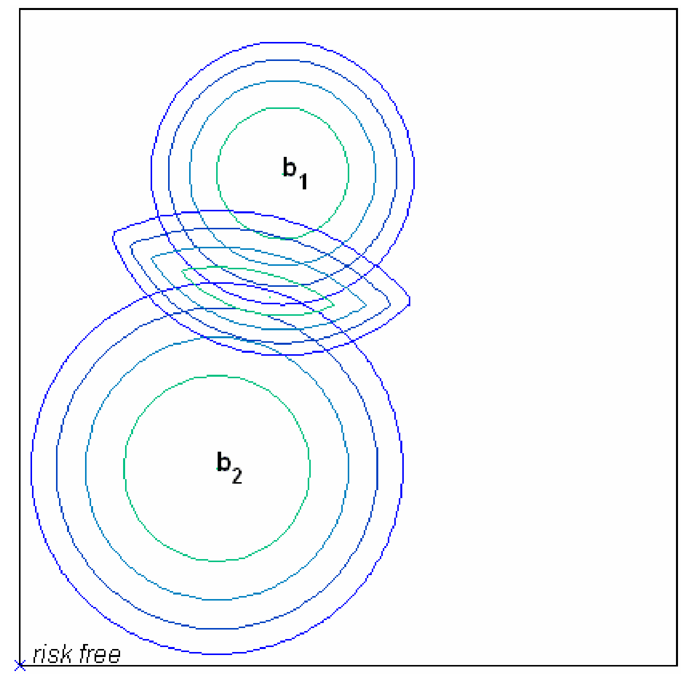
$U_1(p) = \alpha(p) - \lambda_1 TV(p,b_1) - f(p)$

$U_2(p) = \alpha(p) - \lambda_2 TV(p,b_2) - f(p)$

Define the joint utility as

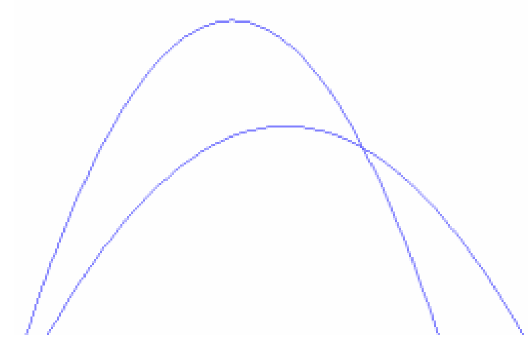
$$U(p) = \min\,[U_1(p), U_2(p)]$$

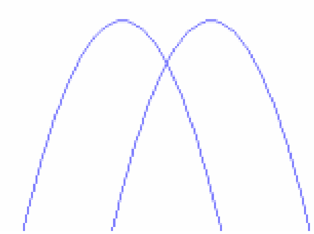
$\Rightarrow$ Don’t need to tweak parameters.



Nice Formulation – Is It Solvable?

- Easily solvable when the λ's are the same.
- Observe that the solution occurs at the maximizer of either utility function or where they are equal.





Algorithm for Solving Pareto

1. Maximize each function separately. If the other function at the maximizer is ≥ the maximized function then done.
2. Maximize either of the functions, limiting the search to the set where the functions are equal

If the λ's are equal this set is just a linear constraint:

$$\underbrace{p^{T}\, Q\left(b_1 - b_2\right)}_{\text{vector}} = \underbrace{\frac{1}{2\lambda}\alpha^{T}\left(b_1 - b_2\right) + \frac{\sigma_{b_1}^{2} - \sigma_{b_2}^{2}}{2}}_{\text{scalar}}$$

If the λ's differ, the set is the surface of an ellipsoid (a hard problem.)

Origin of the "Pareto Constraint"

in the Intersection of Dual Objectives

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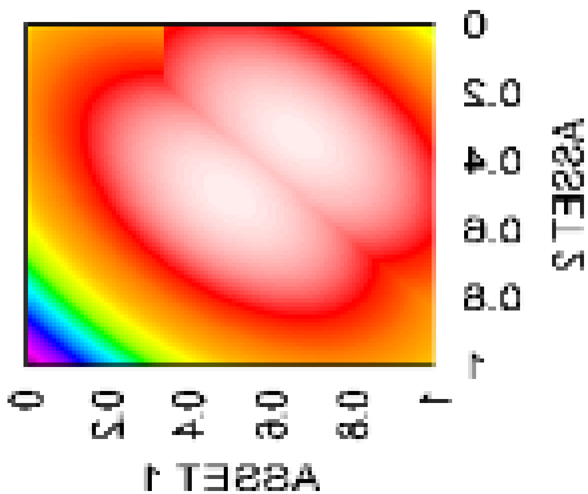

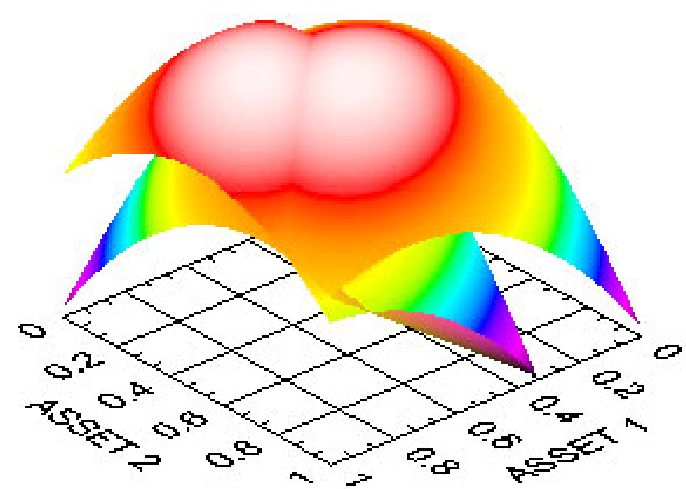

The Pareto Optimal Portfolio

PROPERTIES

- Benchmarks ***b,c*** (***c*** might be cash).
- Guarantee: no other portfolio tracks **both** ***b*** and ***c*** better (*ex ante*, of course).
- Composite benchmark solution **seeks** to track both benchmarks equally by tracking a composite benchmark ***½(b+c)***.
- But generally, ½ is not the right proportion.
- Pareto optimal solution **does** track both benchmarks equally (*ex ante*, of course).
- Equal tracking can be done in either sense:
  - Identical tracking errors, or
  - Identical quadratic utility (risk adjusted return).

Intuition about Pareto Optimum

- Take the case where we demand that the expected risk adjusted returns be equal.

- Our objective, then, is to maximize:

$$\min\left\{(x-b)^{\mathrm{T}} r-\frac{1}{\mathrm{RAP}}(x-b)^{\mathrm{T}} Q(x-b),\ \ (x-c)^{\mathrm{T}} r-\frac{1}{\mathrm{RAP}}(x-c)^{\mathrm{T}} Q(x-c)\right\}$$

- As **RAP**$\rightarrow\infty$, we try to maximize the lesser of the two active returns.

- For a sufficiently aggressive investor, the pareto strategy reduces to tracking whichever benchmark we expect to perform better—no "pareto constraint".

Intuition about Pareto Optimum

2 of 3

- Take the case where ***c*** is cash. The added constraint

$$(b-c)^{\mathrm{T}} Qx = \frac{1}{2}\left[\mathrm{RAP}\cdot(b-c)^{\mathrm{T}} r + b^{\mathrm{T}} Qb - c^{\mathrm{T}} Qc\right]$$

reduces to

$$b^{\mathrm{T}} Qx = \frac{1}{2}(\mathrm{RAP}\cdot b^{\mathrm{T}} r + b^{\mathrm{T}} Qb)$$

- The name of the game:
  Reduce portfolio's covariance with the benchmark by examining assets having low or negative correlation with benchmark.

- We may require this covariance to be as little as half the benchmark variance, when we demand equal tracking errors or we use no α's (dual indexing).

Intuition about Pareto Optimum

- Let $\overline{\text{cov}}, \underline{\text{cov}}$ be the largest, smallest covariance of an individual asset with benchmark ***b***.
- Most aggressive portfolio for which pareto constraint would be employed:

$$\overline{\text{RAP}} = \begin{cases} \dfrac{2\,\overline{\text{cov}} - b^{\text{T}} Q b}{b^{\text{T}} r}, & b^{\text{T}} r > 0 \text{ (bullish on benchmark)} \\ \dfrac{2\underline{\text{cov}} - b^{\text{T}} Q b}{b^{\text{T}} r}, & b^{\text{T}} r > 0 \text{ (bearish on benchmark)} \end{cases}$$

- Amplifies benchmark timing bet.
- Very different set of diversifying assets when benchmark expected to go down—Turnover issue.
- Near $\overline{\text{RAP}}$, may omit “pareto constraint” because it can cause too few names to be included in the basket.

Summary

- Dual benchmarks occur in practice.
- The objectives are a bit ambiguous and can be formulated in different ways.
- In many cases, these problems can be solved by standard portfolio optimizers.