The Option to Rebalance

Michael S. Gamze

The idea of rebalancing is not new. To succeed, any structure has to adjust to new conditions. Every time we do this, we are rebalancing. Quick and intelligent response to changes in the marketplace is a vital feature of successful trading and investment operations. Market microstructure, legal liability, internal policies and business strategies put certain practical restrictions on frequency, scope and nature of rational rebalancing. Higher frequency and wider scope of rebalancing would provide extra flexibility and edge in achieving investment goals, but in the same time they would add to cost of information services, technology and operations. Ability to value the option to rebalance could be critical for rational investment and trading decisions.

Complexity of valuing the option to rebalance is a function of the individual's accepted investment strategy. For the presentation purposes my analysis deals with quantitative investment strategies based on optimization methods. To keep focus on the main topic – a value of the option to rebalance - we would not consider problems related to parameterization of optimization procedures, and we will assume that all statistical properties of asset dynamics are known.

The presentation consists primary of three related sections and conclusions. The first section covers basic of single-step (static) and multi-step (dynamic) investment processes, and discusses their fundamental differences. It also refers to optimization models employed in static and dynamic investment products. The second sections puts these investment models in the framework of stochastic dynamic optimization theory, and provides a formal definition for the value of the option to rebalance in mathematical terms. The third section provides examples – found in ordinary investment situations - of the key concept and properties of the option to rebalance.

Finally, conclusions summarize results, discuss areas of their applicability and further research.

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The Option to Rebalance

- Has a positive net present value
- Effects our initial asset allocation
- Leads to higher expected return
- Converts volatility in opportunity

Focus:

- Quantitative investment products

Out of consideration:

- Parametrization of quantitative procedures
- Selection of utility functions

Contents

- Single-step and multi-step investment process
- Stochastic dynamic optimization and formal definition of option to rebalance
- Simple analytical results

Single-Step Investment Process

Principle:

Risk reduction through diversification

or

"Don't put all your eggs in one basket"

Diversification

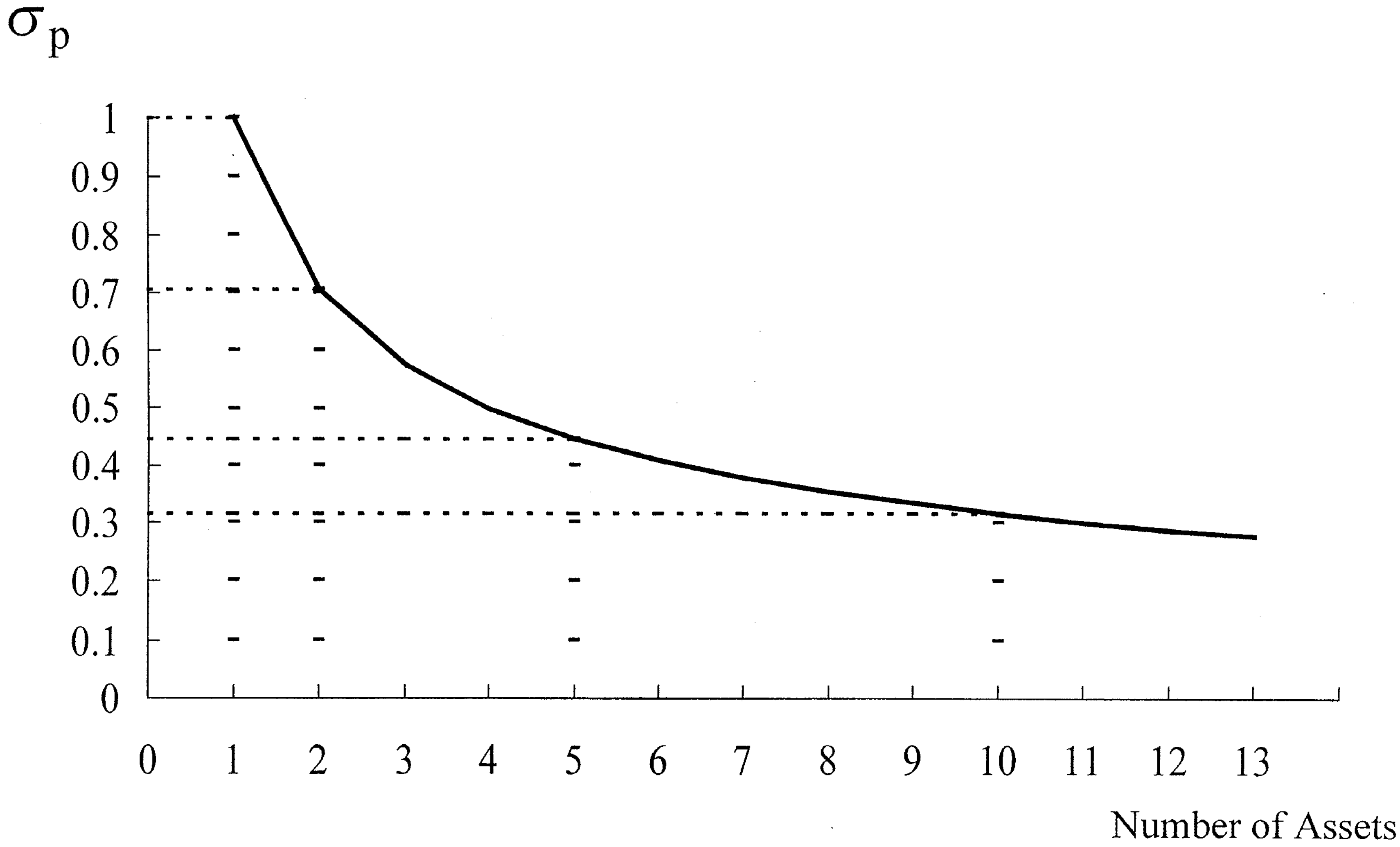
n – number of assets

m – expected return

σ – variance

$$r_p = \frac{1}{n}\sum_{i=1}^{n} r_i = m$$

$$\sigma_p^2 = \mathrm{var}(r_p) = \frac{1}{n^2}\sum_{i=1}^{n} \sigma_i^2 = \frac{\sigma^2}{n}$$


σp
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8 9 10 11 12 13
Number of Assets

Markowitz Optimization

n – number of assets

$\bar{r}$ – expected return

σ_{ij} – covariance

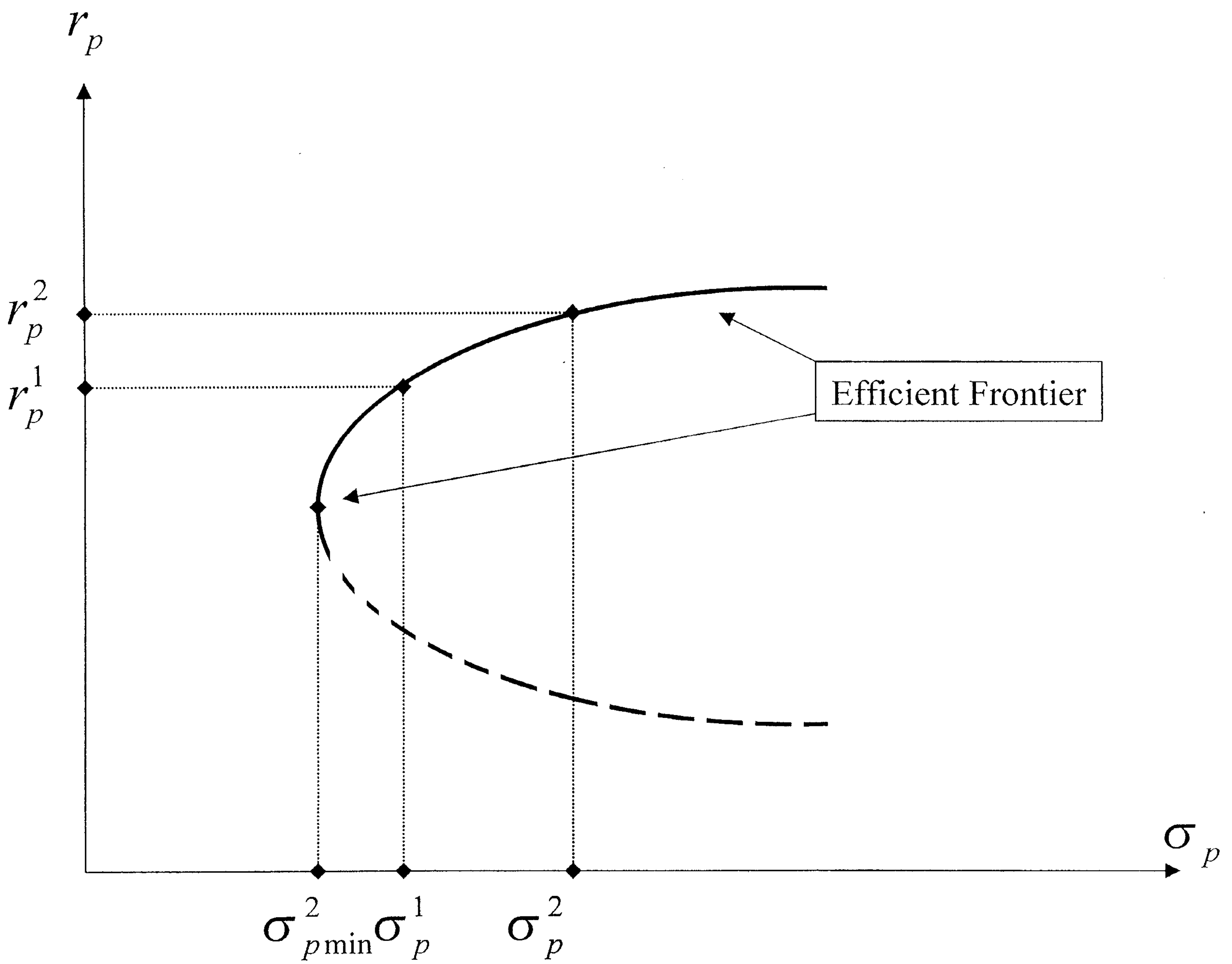
w_i – weight in portfolio

$$\min \quad \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

subject to

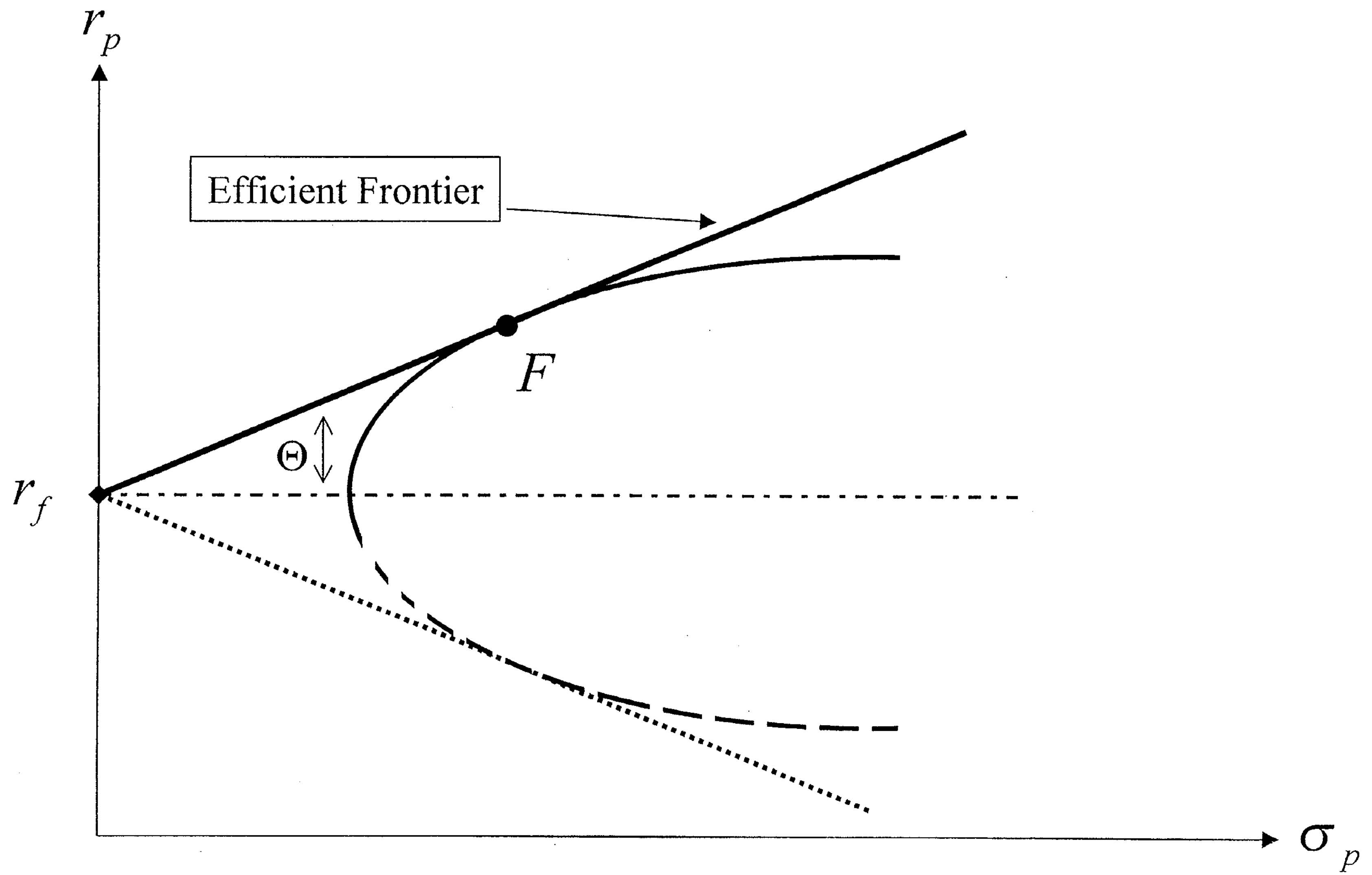
$$\sum_{i=1}^{n} w_i \overline{r_i} = r$$

$$\sum_{i=1}^{n} w_i = 1$$



r_p
r_p^2
r_p^1
Efficient Frontier
σ_p
$\sigma^2_{p\min}$
σ_p^1
σ_p^2

Portfolio with Risk Free Asset



$$\tan \theta = \frac{\bar{r}_p - r}{\sigma_p}$$

F – maximize $\tan \theta$ on feasible set

$$\max \frac{\sum_{i=1}^{n} w_i \left(\bar{r}_i - r_f \right)}{\left(\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \right)^{1/2}}$$

subject to

$$\sum_{i=1}^{n} w_i = 1$$

Multi-Period Investment Process

Principle:

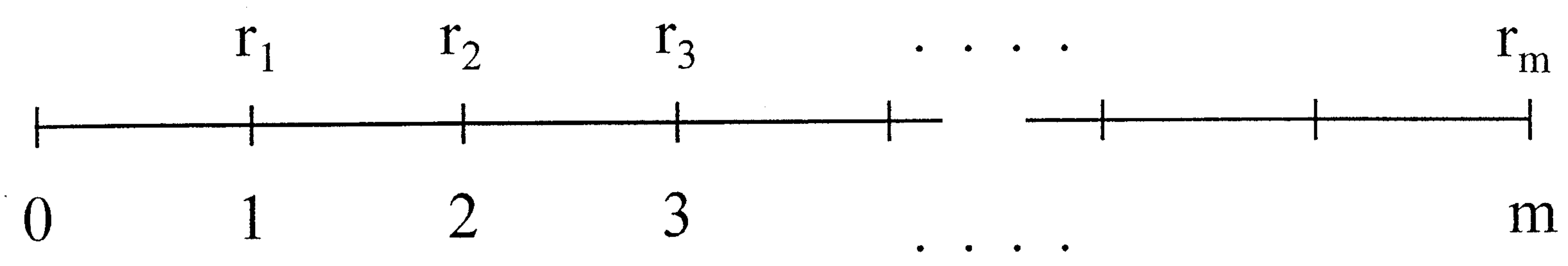
Volatility is opportunity

or

"Buy low and sell high"

Geometric Mean (Log Utility) Approach

- Natural m-period generalization of Markowitz efficiency
- Consistent with many institutional investment mandates (pension funds, trusts)
- Consistent with Black-Scholes pricing (important when the same portfolio has options and underlying assets)
- Analytical tractability



$$P_m = P_0 \cdot r_1 \cdot r_2 \cdot \ldots r_m$$

$$\bar{r} = \left(r_1 \ldots r_m\right)^{1/m}$$

$$P_m = P_0 \cdot \bar{r}^{\,m}$$

n - number of assets

$$\Delta p_i = \mu \cdot p_i \Delta t + p_i \Delta z_i$$

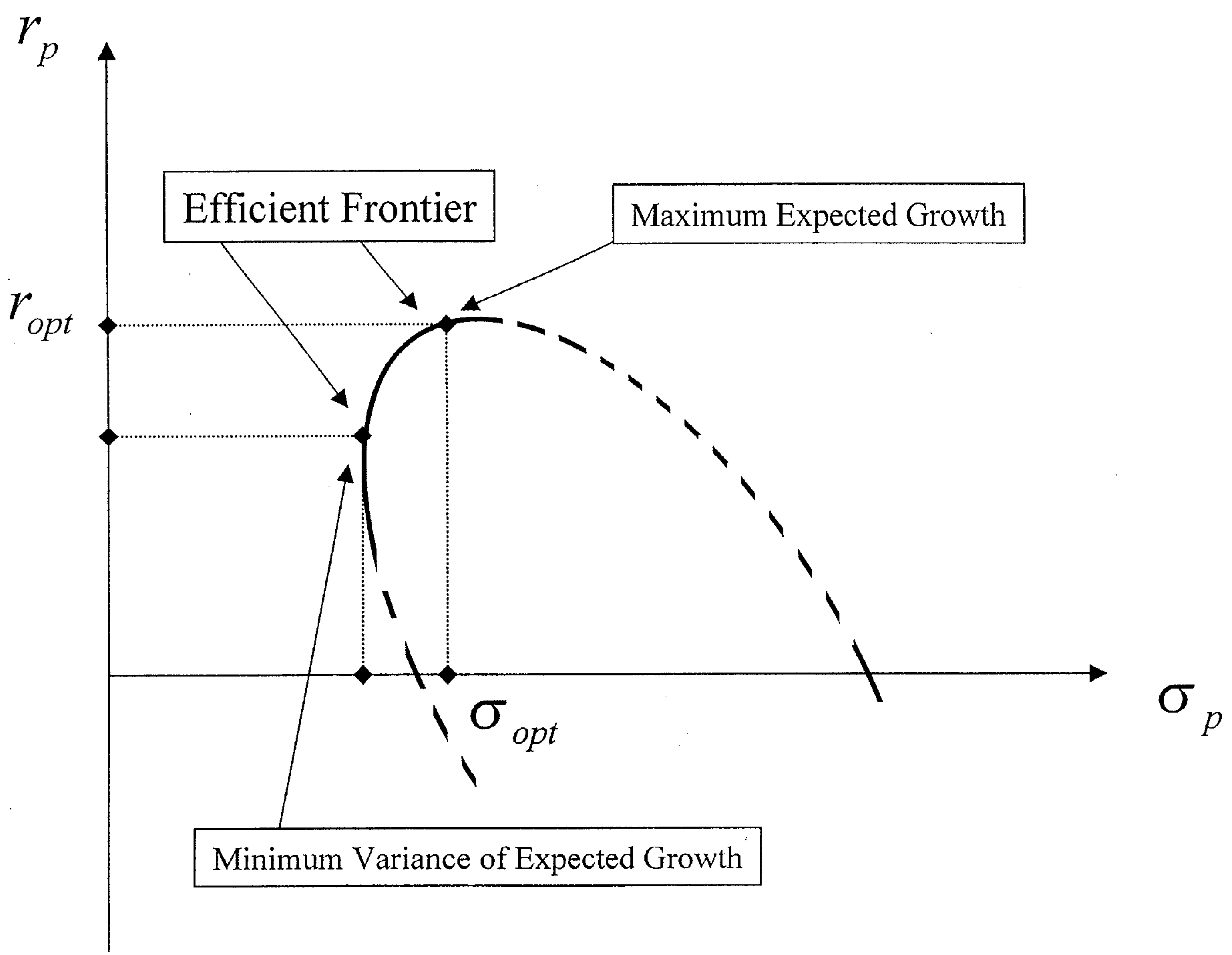
z_i – Wiener process with variance σ_1

$$\text{cov}(\Delta z_i, \Delta z_j) = E(\Delta z_i, \Delta z_j) = \sigma_{ij} \Delta t$$

$$\max \sum_{i}^{n} w_i \mu_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \sigma_{ij} w_j$$

subject to:

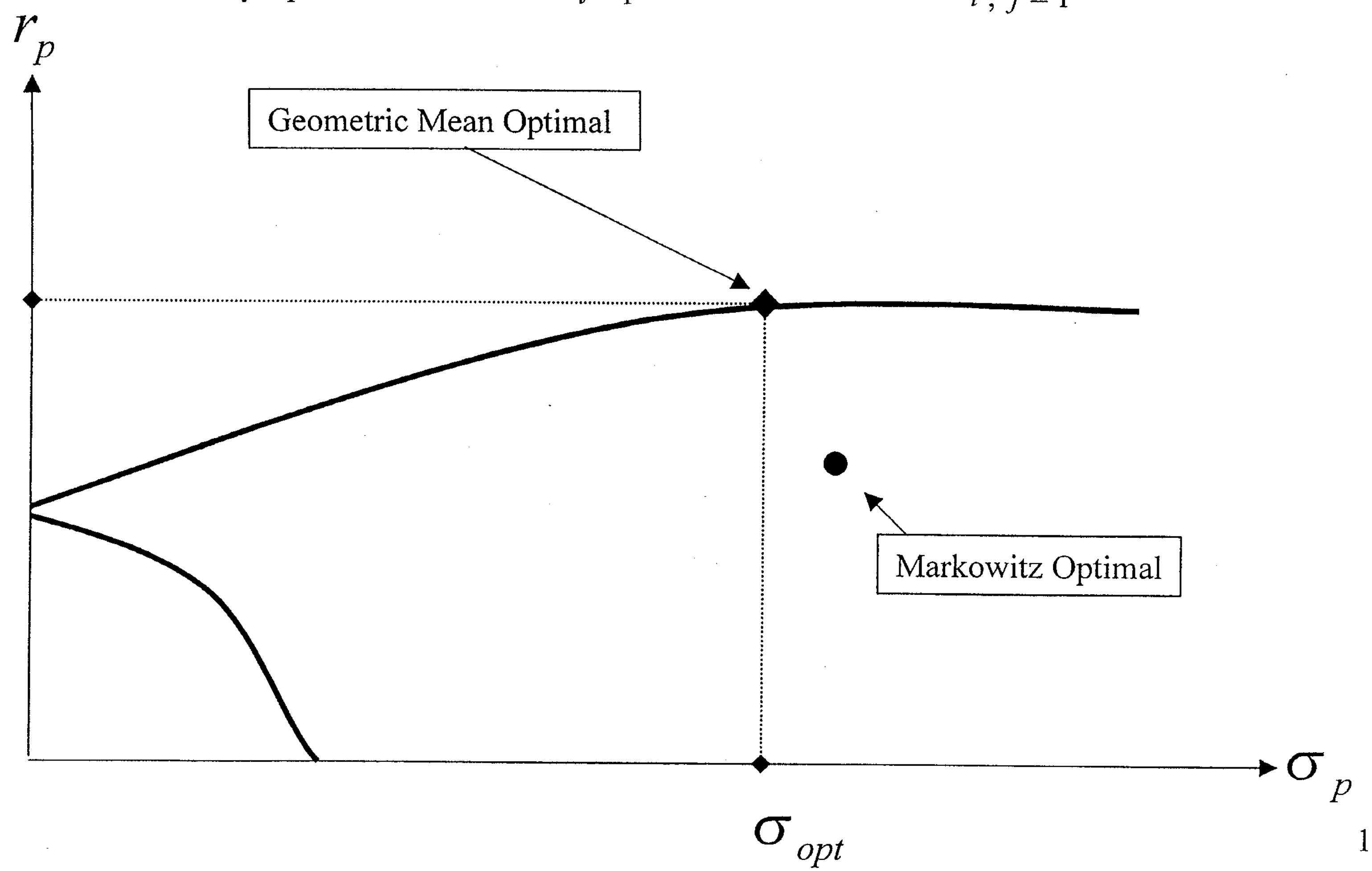
$$\sum_{i=1}^{n} w_i = 1$$



r_p
r_opt
Efficient Frontier
Maximum Expected Growth
σ_opt
σ_p
Minimum Variance of Expected Growth

Risk Free Asset

$$\max \quad [(1-\sum_{i=1}^{n} w_i)r_f + \sum_{i=1}^{n}(\mu_i w_i - \frac{1}{2}\sum_{i,j=1}^{n} w_i \sigma_{ij} \cdot w_j)]$$



Stochastic Optimization

(*informational structure*)

1 How a system processes new information

- *single-step (static) stochastic optimization: solution is not effected in the future*
- *multi-step (dynamic) stochastic optimization: solutions is corrected based on new information*

2 Timing between arrival of new information and decision

- *solution is deterministic vector if in each period of time we have access to new information before decision*
- *solution is a function of new observations*

Stochastic Optimization

(*mathematical classification*)

1 Form of objective function

- *P - models*
- *V - models*
- *E - models*

2 Character of constraints

- *firm*
- *probabalistic*
- *statistical*

3 Form of solution

- *deterministic*
- *solution rule*

Markowitz Optimization

$$\min V[(w, \bar{r})]$$

subject to

$$E[(w, \bar{r})] = \mu$$

$$(1, w) = 1$$

Two-Step Stochastic Optimization

(*problems with rebalance*)

$$\min_{(w_1, w_2(\xi))} E\{\psi_0(\xi, w_1, w_2(\xi))\}$$

subject to

$$E\{\varphi_k(\xi, w_1, w_2(\xi)\} \le \alpha_k, k \in K$$

$$w_1 \in G_1$$

$$w_2 \in G_2$$

Additive Objective Functions

$$\max_{(w_1, w_2(\xi))} E[\psi_0(\xi, w_1) + \varphi_0(w_2(\xi))]$$

subject to

$$P\{h(w_2(\xi)) \leq b(\xi) - g(\xi, w_1)\} \leq \alpha;$$

$$\varphi(w_1) \leq 0$$

Solution - $(w_1^*, w_2^*(\xi))$

Option to Rebalance

Two-step problem

$$C^{(1)} = E\{u(w_1^*, w_2^*(\xi)] - E[u(w^*)]$$

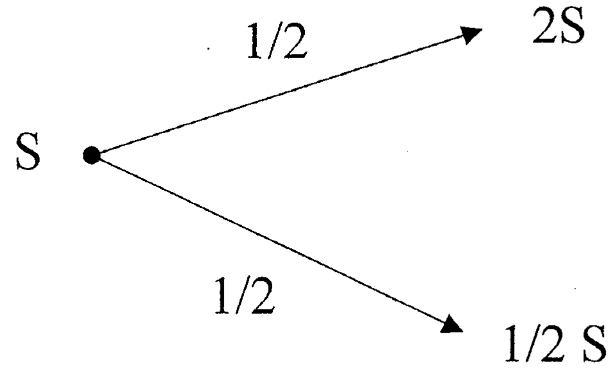
Multi-step problem

$$C^{(n-1)} = E[u(w_1^*, w_2^*(\xi), \ldots, w_n^*(\xi)] - E[u(w^*)]$$

Example 1: Log utility, two assets.

I. Risk free asset with $r_f = 0$

II.



For both I and II, growth rates are zero.

Strategy: 50/50 split between I and II.

$$w_1 = w_2 = \frac{1}{2} \ \forall \ n = 1, \ldots, m$$

Expected growth for this strategy:

$$\nu = \frac{1}{2}\ln\left[\frac{\frac{1}{2}+\frac{1}{2}\cdot 2}{\frac{1}{2}+\frac{1}{2}}\right] + \frac{1}{2}\ln\left[\frac{\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}}\right] =$$

$$= \frac{1}{2}\ln\frac{5}{8} \approx .0589 = 5.89\ \%$$

What is an optimum strategy?

$$\mu_2 - r_f - \sigma_2^2 w_2 = 0 \qquad \mu_2 = \nu_2 + \frac{\sigma_2^2}{2} = \frac{\sigma_2^2}{2}$$

$$\frac{\sigma_2^2}{2} = \sigma_2^2 w_2 \qquad w_2 = \frac{1}{2}$$

What is the value of rebalancing?

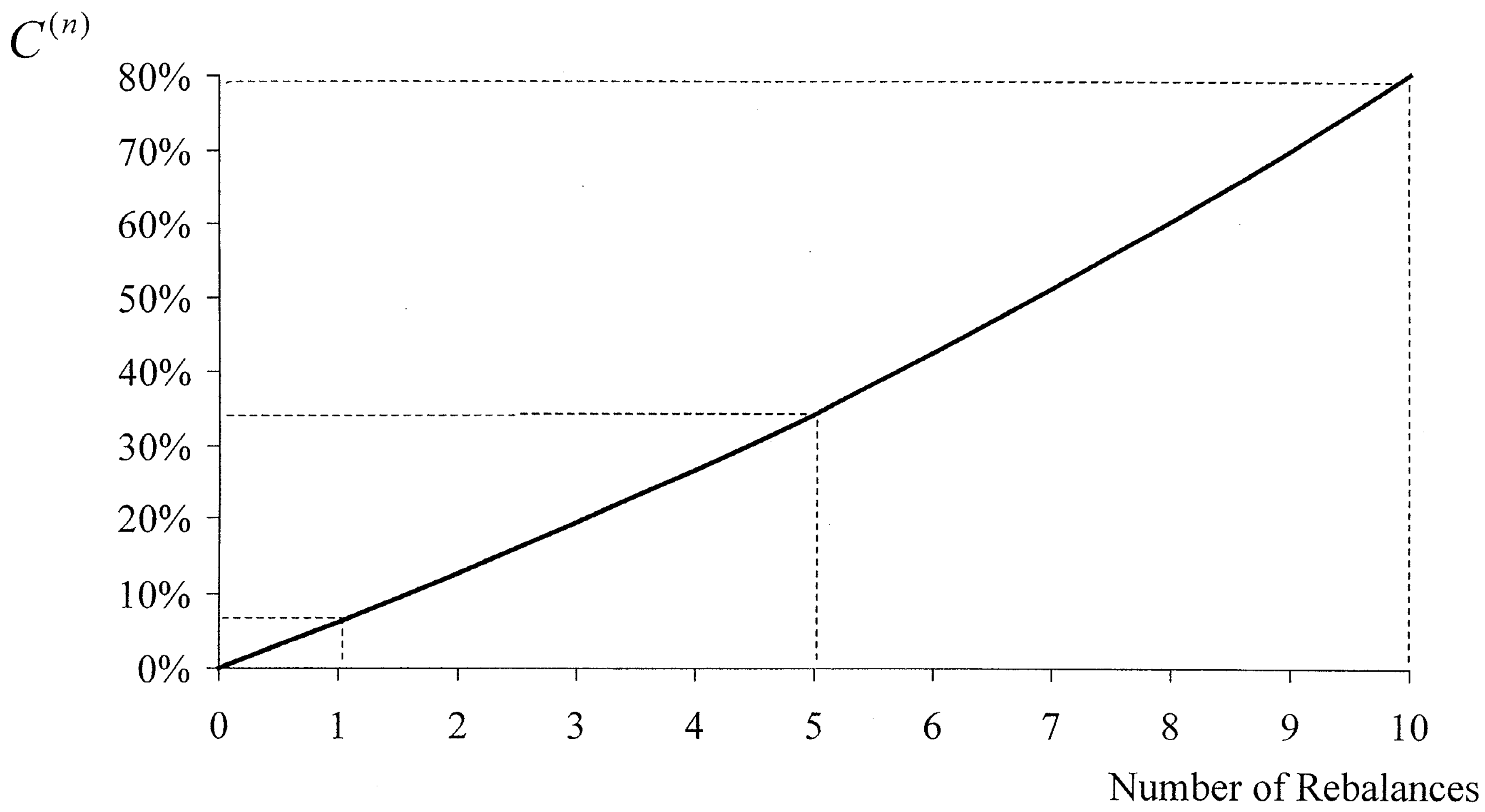
$$C^{(n)} = P_0(e^{vn} - 1)$$

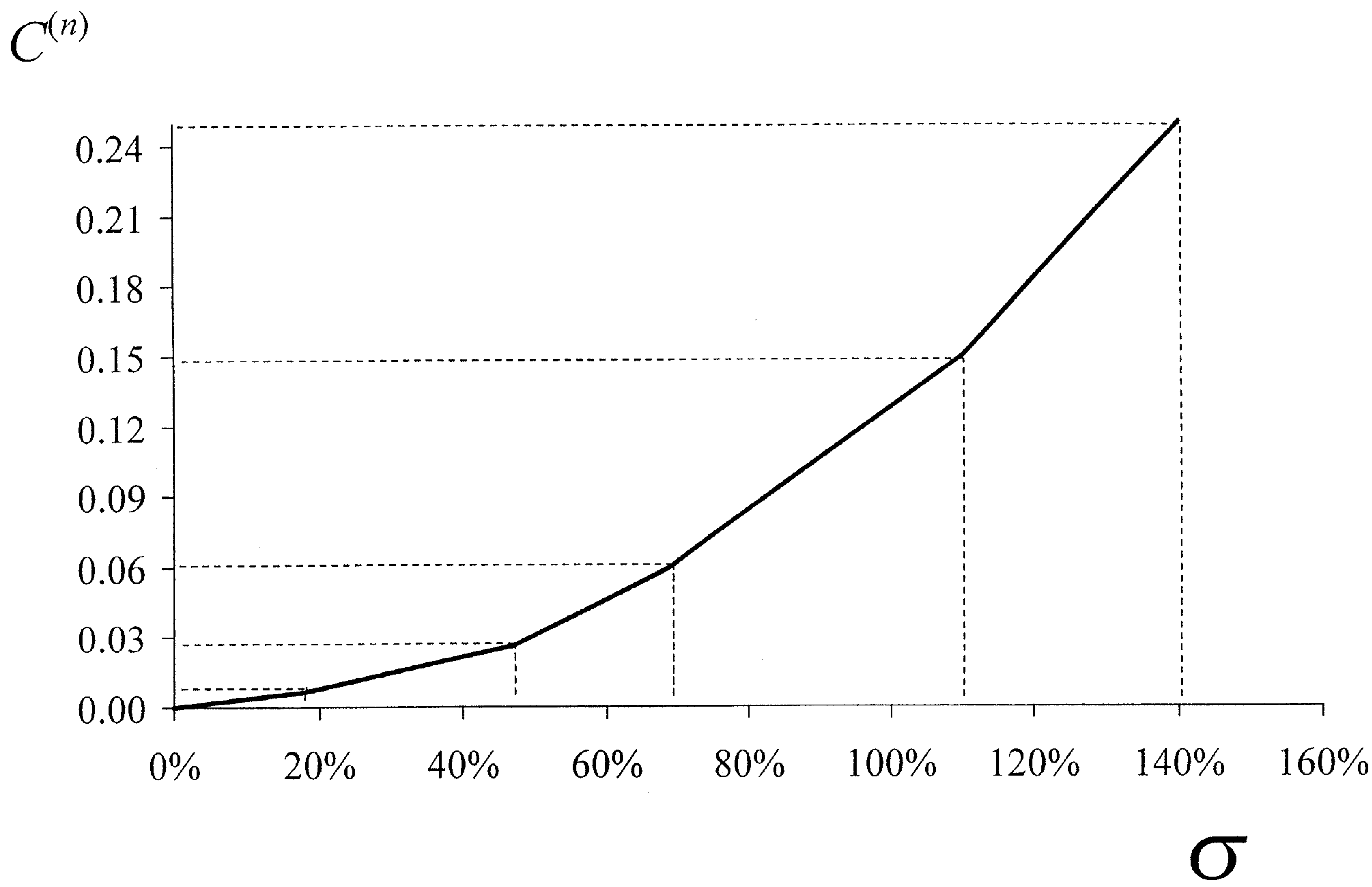
for our example

$$C^{(1)} \approx 1.06 - 1 = 6\%$$

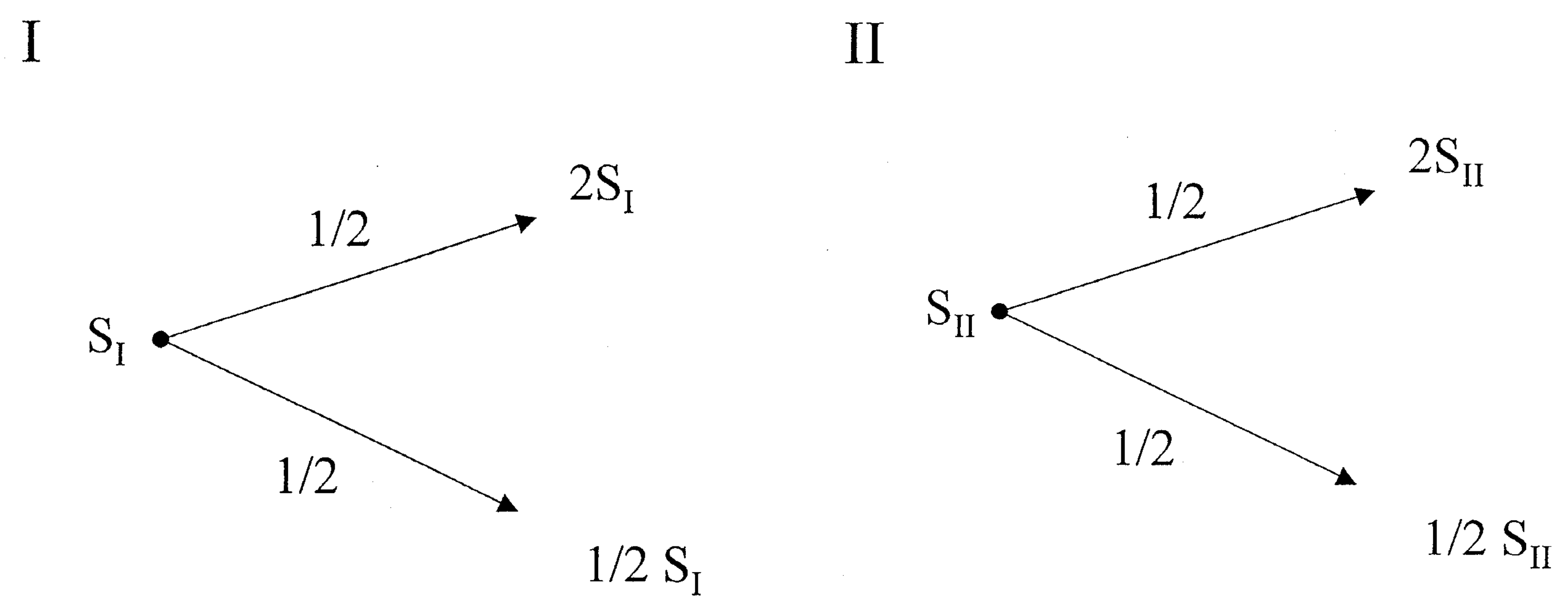
$$C^{(5)} \approx 1.34 - 1 = 34\%$$

$$C^{(10)} \approx 1.80 - 1 = 80\%$$


C(n)
80%
70%
60%
50%
40%
30%
20%
10%
0%
0
1
2
3
4
5
6
7
8
9
10
Number of Rebalances


$C^{(n)}$
0.24
0.21
0.18
0.15
0.12
0.09
0.06
0.03
0.00
0%
20%
40%
60%
80%
100%
120%
140%
160%
σ

<u>Example 2:</u> Log utility, two risky assets.



For I and II, growth rates are zero.
<u>Strategy:</u> 50/50 split between I and II.

$$w_1 = w_2 = \frac{1}{2} \ \forall \ n = 1, \dots, m$$

Expected growth rate:

$$v = \frac{1}{4}\ln[\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 2] + \frac{1}{2}\ln[\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot\frac{1}{2}] + \frac{1}{4}\ln[\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}]$$
$$= \frac{1}{2}\ln\frac{5}{4} \approx .1116 = 11.16\%$$

Optimum strategy:

$$w_1 = w_2 = \frac{1}{2}$$