Optimal Turnover and the Single Period Assumption in Portfolio Optimization

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This Presentation in Outline

- Review the main four sources of uncertainty in mean-variance optimization
- Consider how to encompass all sources of uncertainty by augmenting the objective function
- Focus the discussion on a "structural" form of estimation error that has been relatively less explored in the literature
  - the "single period" assumption built into the Markowitz mean-variance optimization process
- Introduce a "rule of thumb" into traditional MVO to produce more efficient tradeoffs between risk-adjusted returns and transaction costs

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We Walk From Where We Stand

- Markowitz and Levy (1979) propose a function of mean variance as a representation of investor utility

  $U = R - S^2 / T$

- This is just risk adjusted return where the size of the risk penalty can be scaled to fit the investor's risk tolerance

- Assumptions

  - The parameters of return distributions are known with certainty. In the real world, we can get it wrong.

  - The future is one long period in which our input parameters values (that are both correct and certain) never change.

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Four Sources of Uncertainty

- We assert there are four main sources of uncertainty in an optimization
  - Market and security risks (the kind that risk models are meant to deal with)
  - Parameter estimation error in the return estimates
  - Parameter estimation error in the risk estimates
  - The single period framework that is assumed for MVO
- If you consider that each of these effects can interact with the others
  - you have a 4*4 matrix of sixteen things
  - the matrix is symmetric about the diagonal so there are really ten terms we have to worry about, of which traditional optimization considers only one
  - If you think trading costs are uncertain, now you have a 5 * 5 matrix with fifteen terms
- To my knowledge no one has tried dealing with all ten

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More Thoughts on Single Period

- In the real world, things change and our parameter estimates for return and risk (even if initially exactly correct) are likely to change as well.

- If transaction costs are zero, we can simply adjust our portfolio composition to optimally reflect our new beliefs whenever they change.

- If transaction costs are not free, the single period assumption is a serious problem.

- If transaction costs are large (e.g. capital gain taxes), the single period assumption is wholly unrealistic.   Tax authorities also seem to be interested in things like weeks, months and especially "tax years."

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Multi-period Optimization

- Mossin (1968) suggests an explicit multi-period formulation for portfolio optimization

- Cargill and Meyer (1987) focus on the risk side of the multi-period problem

- Merton (1990) introduces continuous time analog to MVO

- Pliska (1997) provides a discrete time analog to MVO

- Li and Ng (2000) provide a framework for multi-period MVO using dynamic programming

- Sneddon (2005) provides a closed form solution to the multi-period optimization *including optimal turnover*

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Multi-period Optimization, cont'd

- Parameter estimation error is the killer here
  - In practice, investors have enough difficulty estimating risk and return parameters as of "now"
  - *For multi-period optimization we must estimate now what the parameters will be for all future periods*

Smart Rebalancing

- Numerous "smart" rebalancing rules have been proposed to avoid trading costs when the expected improvement is not significant

- Rubenstein (1991) examines the efficiency of continuous rebalancing and proposes a rule for avoiding spurious turnover

- Kroner and Sultan (1993) propose a "hurdle" rule for rebalancing currency hedges when return distributions are time varying

- Engle, Mezrich and Yu (1998) propose a hurdle on alpha improvement as the trigger for rebalancing

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Even Smarter Rebalancing

- Bey, Burgess, Cook (1990) use bootstrap resampling to identify "indifference" regions, along a fuzzy efficient frontier

- Michaud (1998) uses resampling to measure the confidence interval on portfolio return and risk to form a "when to trade rule". Elaborated upon in Michaud and Michaud (2002) and patented.

- Markowitz and Van Dijk (2003) propose a rebalancing rule designed to approximate multi-period optimization, but argue it is mathematically intractable (at least in closed form)

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Single Period Model with Costs

- Grinold and Stuckelman (1993) consider a value added/turnover efficient frontier. They derive that under certain common assumptions, value added is approximately a square root function of turnover

- Common practice is extend the objective function to include transaction costs (C) that are linearly amortized at an period rate (A) that reflects the expected economic life of the benefits of the transaction

$$U = R - S^2 / T - (C \times A)$$

- The expected average holding period for the positions resulting from a transaction is just the reciprocal of the expected one-way turnover

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Geometric Versus Linear Tradeoffs

- For small transaction costs, arithmetic amortization is sufficient, but if costs are large we need to consider compounding

- Assume a trade with 20% trading cost and an expected holding period of one year.

  - We can get an expected alpha improvement of 20%. But if we give up 20% of our money now, and invest at 20%, we only end up with 96% of the money we have now.

- Solution is to adjust the amortization rate to reflect the correct geometric rate

Lets Define The Probability of Realization

- We define the probability of realization, P, like a one-tailed T test

$$P = N(((U_o - U_i) / TE_{io}) * (1/A)^{.5})$$

- $N(x)$ is the cumulative normal function:

$$N(x) = \frac{1}{\sqrt{2p}} \int_{-\infty}^{x} e^{-u^2/2} \, du$$

The Realization Probability

- The numerator is the improvement in risk adjusted return between the optimal and initial portfolios
- The denominator is the tracking error between the optimal and initial portfolios. Essentially it's the standard error on the expected improvement in utility
  - If there is no tracking error between the initial and optimal portfolios, P approaches 100%. Consider "optimizing a portfolio" by getting the manager to cut fees. The improvement in utility is certain no matter how short the time horizon.
  - *Not something to which we usually pay attention*
- If turnover is very low, A will approach zero, so P will approach 100%. For long time horizons, we have the classical case that assumes certainty

Some Possible Uses for "Probability of Realization"

- Create an indifference trading rule
  - Optimize but don't trade at all if P is less than some threshold value that you consider statistically significant
- Create a "hurdle" trading rule
  - Optimize but don't trade at all if the probability of realization weighted value-added (P * value-added) is less than some threshold value
- Pick your spot on the "turnover/value-added frontier
  - Such that P is maximized
  - Such that probability of realization weighted value-added (P * value-added) is maximized
- Use P to adjust the amortization rate so your optimization gets closer to the optimal amount of turnover

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Empirical Example

- Initial Portfolio 85 Large Stocks
- S&P 500 Benchmark
- Random Z-scores as alphas
- Risk Tolerance = 40
- Expected Turnover 25% per annum
- No position bigger than 3%
- No position smaller than .25%
- 20 cents per share trading costs

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Probability of Realization

Probability of Realization
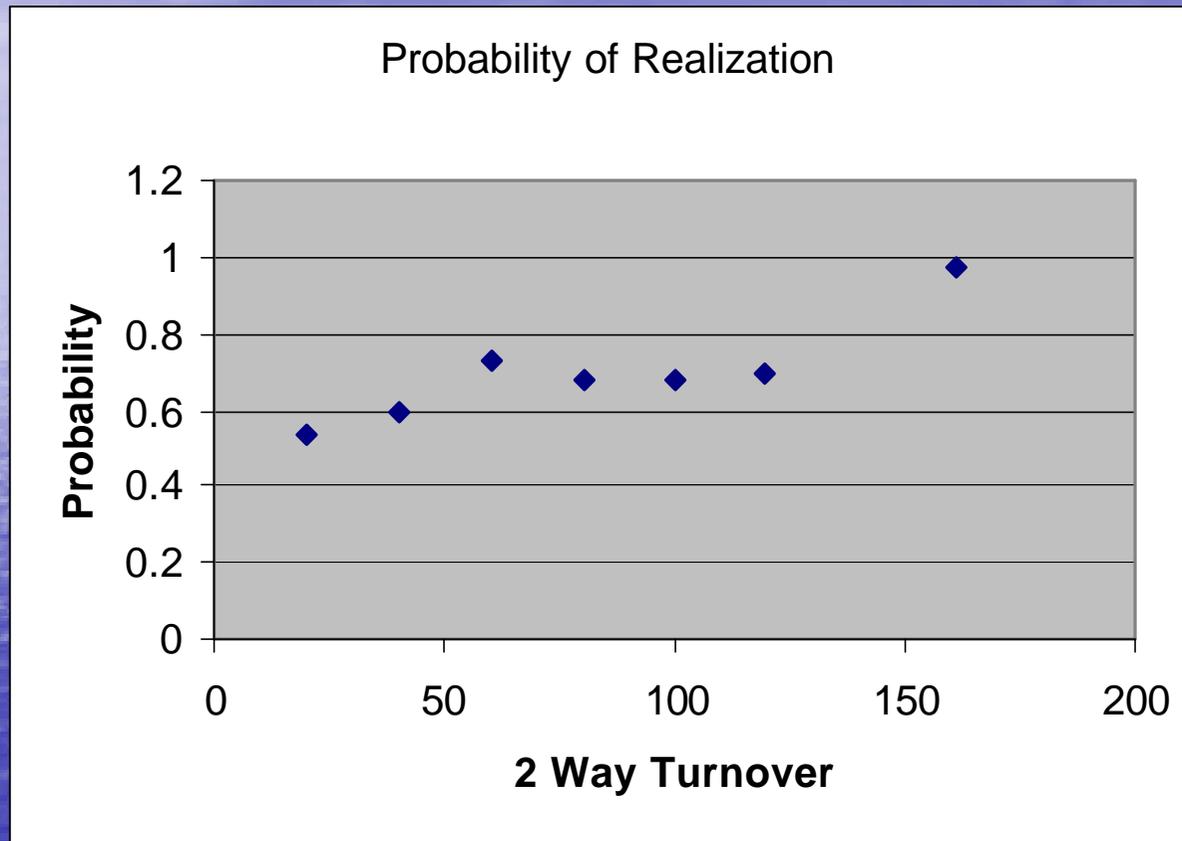
A Very Approximate Fix

- Even if we are amortizing our costs sensibly, we are still maximizing the objective function to directly trade a unit of risk adjusted return for a unit of amortized cost per unit time.

- This is only appropriate if we are certain to realize the economic benefit of the improvement in risk adjusted return, which is only true over an infinite time horizon

- We propose to adjust the amortization rate to reflect the probability of actually realizing the improvement in utility over the expected time horizon, and the investor's aversion to the uncertainty of realization

$$U = R - S^2 / T - (C \times \Gamma)$$

$$\Gamma = A / (1- L * (1-P)), \quad L = 1 - (T/200)$$

P is the probability of realizing the improvement in risk adjusted return over the expected time horizon and L is the range of (0,1)

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One Last Idea

- Another approach would be to add the uncertainty of realization to the objective function

$$U = R - S_b{}^2 / T_b - (C \times A) - (A * TE_{IO}{}^2 / T_{IO})$$

- Wang (1999) provides an analytical solution

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Conclusions

- The single period assumption in MVO implies that trading costs and improvements in utility can be traded as if both are certain
- In addition to other sources of estimation error, finite holding periods imply that the improvement in utility is uncertain
- We must therefore consider the probability of realizing an improvement in utility arising from an optimization as being between 50% and 100%
- The tracking error between the initial portfolio and the optimal portfolio is an approximation of the standard error of the increase in utility, so probability of realization can be computed.

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