Imperial College London

TANAKA BUSINESS SCHOOL

Portfolio Construction in a Regression Framework

James Sefton

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Motivation

- The approach can be used to answer the following questions:
  - Are the weights in my portfolio the result of one or two abnormal events?
  - How well are my portfolio weights estimated? Are some positions critical to the portfolio performance?
  - Are my holdings sub-optimal? Is the evidence strong enough to warrant a trade?
  - How different are two portfolios? Is one more efficient than the other? Or are they both efficient but just offer different risk-return trade-offs?
  - Do I have significant tilts in my portfolio? Are these a result of stock picks or style tilts?
  - What are the costs in terms of portfolio efficiency of my investment constraints?
  - Is the new information sufficient to warrant a rebalancing?

What is so different about this approach?

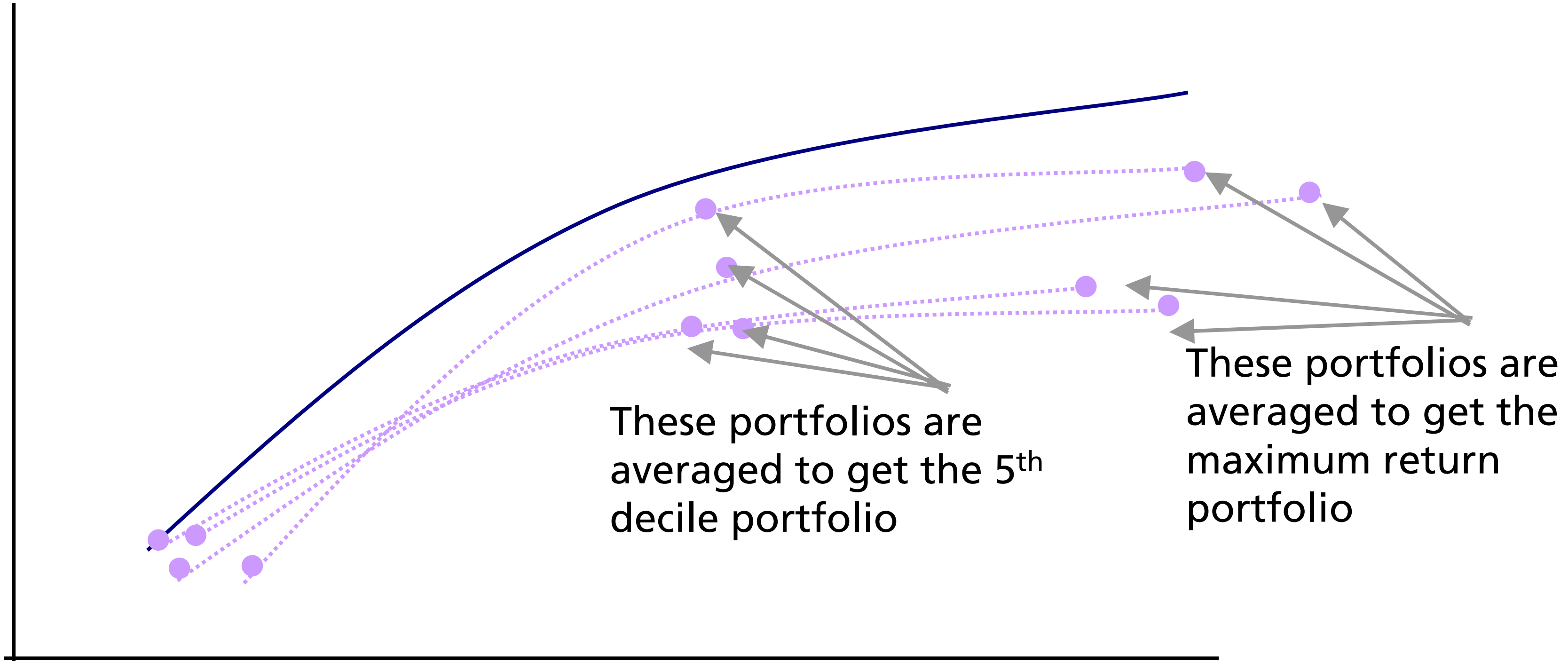
- *Modified* Mean-Variance Problem: Given a time series of returns to a set of assets, estimate portfolio weights that maximise portfolio return for a given level of risk.
  - Step 1. Assume normality, and estimate from the time series data, the mean and covariance matrix of asset returns.
  - Step 2. Using an optimiser, estimate the optimal portfolio weights.
- Many authors (Jobson and Korkie, 1981; Jorian, 1992; Broadie, 1993; Michaud, 1989, Best and Grauer, 1991) have noted how sensitive the portfolio weights in Step 2 are to sampling errors in Step 1. Michaud coined the term that optimisers are '*error maximisers'.*
- The regression approach is 1 step procedure from data to portfolio. We can use all the developed regression diagnostics to analyse our portfolio.

Resampled Efficient Portfolios

- Michaud proposed an *ad hoc* procedure to limit the sampling error problem that has received considerable attention. It depends crucially on imposing a no-shorting constraint on all estimated portfolios.
  - Step 1. From the observed asset returns series, estimate a mean vector, μ_0, and covariance matrix V_0 of the asset returns.
  - Step 2. Using the estimated distribution, generate another set of asset returns and re-estimate a new mean vector, μ_i, and covariance matrix V_i of the generated asset return series.
  - Step 3. Estimate a minimum variance and maximum return portfolio from *the* distribution $N(\mu_i, V_i)$. Also calculate the other 8 ranked decile portfolios with volatility equally spaced between the minimum and maximum.
  - Step 4. Go to step 2 and repeat *n times.*
  - Step 5. The 10 resampled efficient ranked portfolios are then the simply the average of the *n* estimated decile portfolios.

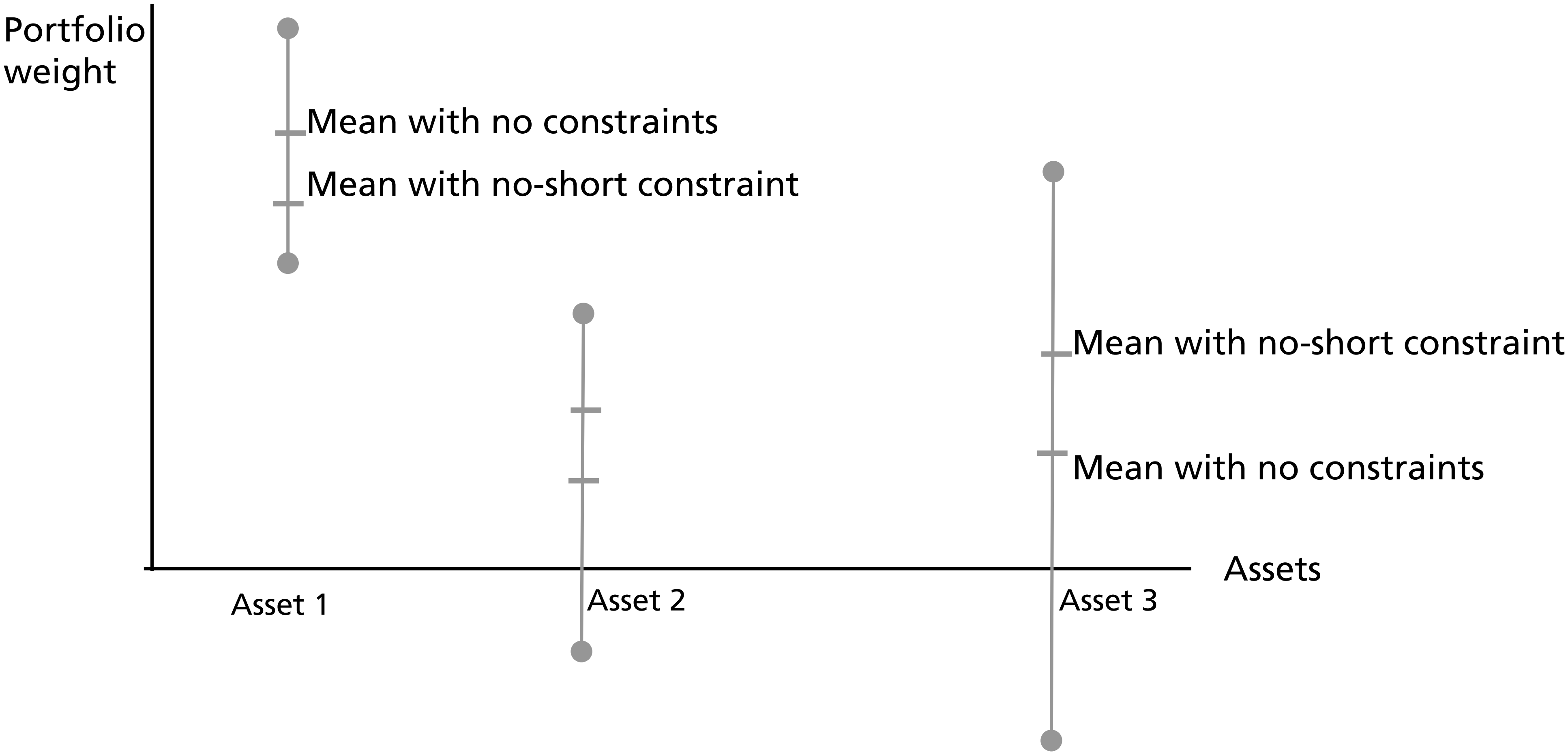
Resampled Frontiers

- The resampled frontiers all lie below the estimated frontier estimated form μ_0 and V_0.



Resampling is a shrinkage estimator – Scherer (2004)

- The no-short constraint means that
  - average weights of low return/high volatility assets are biased up
  - average weights of high return/low volatility assets are biased down



The Regression Approach –Britten-Jones (1999)

- The issue is not the magnitude of the estimation error *per se* but rather the way it impacts upon the portfolio construction process – Broadie (1993).
- The advantages of a regression based approach is that
  1. It effectively maps returns directly into portfolio weights. A one step procedure!
  2. It is statistically rigorous and has a long pedigree!
  3. There is a substantial toolkit that can be brought to bear on regression problems.
  4. As it is simple, it can be easily extended to look at many practical problems e.g. incorporating priors, forecasts, sensitivity analysis.

The Regression Framework

- Define the matrices of asset returns and indicators as

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1N} \\ r_{21} & r_{22} & \cdots & r_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ r_{T1} & r_{T2} & \cdots & r_{TN} \end{bmatrix} \in \mathbb{R}^{T \times N} \qquad e_T = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{T} \qquad e_N = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{N}$$

- And the portfolio weights *were* the least squares estimate in the regressions

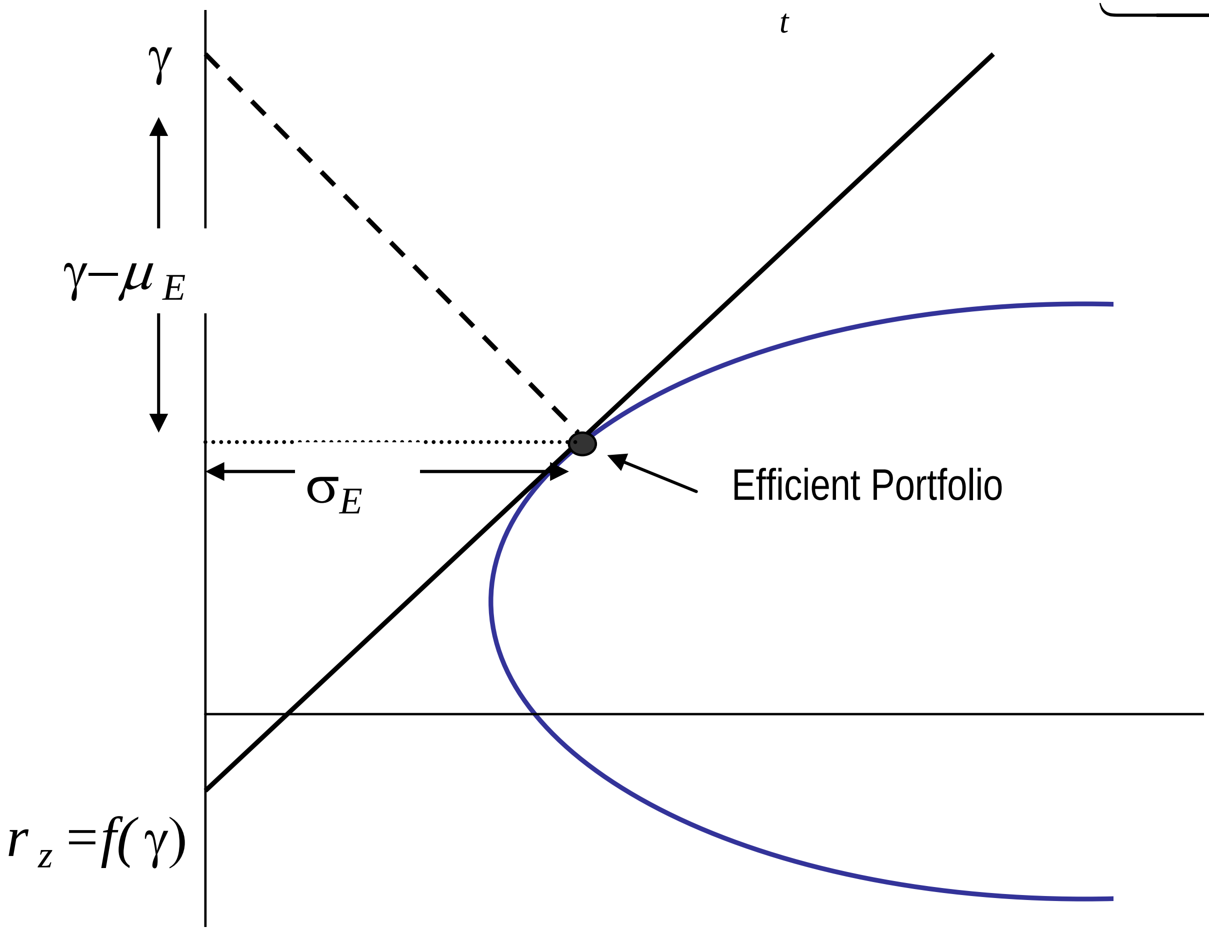
$$\gamma e_T = Rw + \varepsilon$$

For some constant γ subject to the restriction that $e_N{}' w = 1$

The Regression Framework

- As γ varies the regression weights trace out the portfolios on the efficient frontier.

$$\varepsilon'\varepsilon = \sum_t (\gamma - \mu_E)^2 + \underbrace{w'(R-\mu)'(R-\mu)w}_{\sigma_E^2}$$



An Asset Allocation Example

- To illustrate the technique we build a portfolio of six equity funds of six markets. The returns to the funds are the returns to the relevant Dow Jones Global Index between Oct-96 and Sep-06.

- A summary of the fund returns is

	Annualised Mean Return	Annualised Volatilities	Correlation Coefficients US	Canada	UK	Euro Area	Japan	Asia Pacific
US	9.6	17.4	1.00	0.74	0.65	0.69	0.34	0.53
Canada	14.3	18.5		1.00	0.58	0.66	0.42	0.61
UK	10.6	16.4			1.00	0.83	0.34	0.53
Euro Area	12.2	20.4				1.00	0.38	0.58
Japan	3.9	22.6					1.00	0.46
Asia Pacific	6.0	20.1						1.00

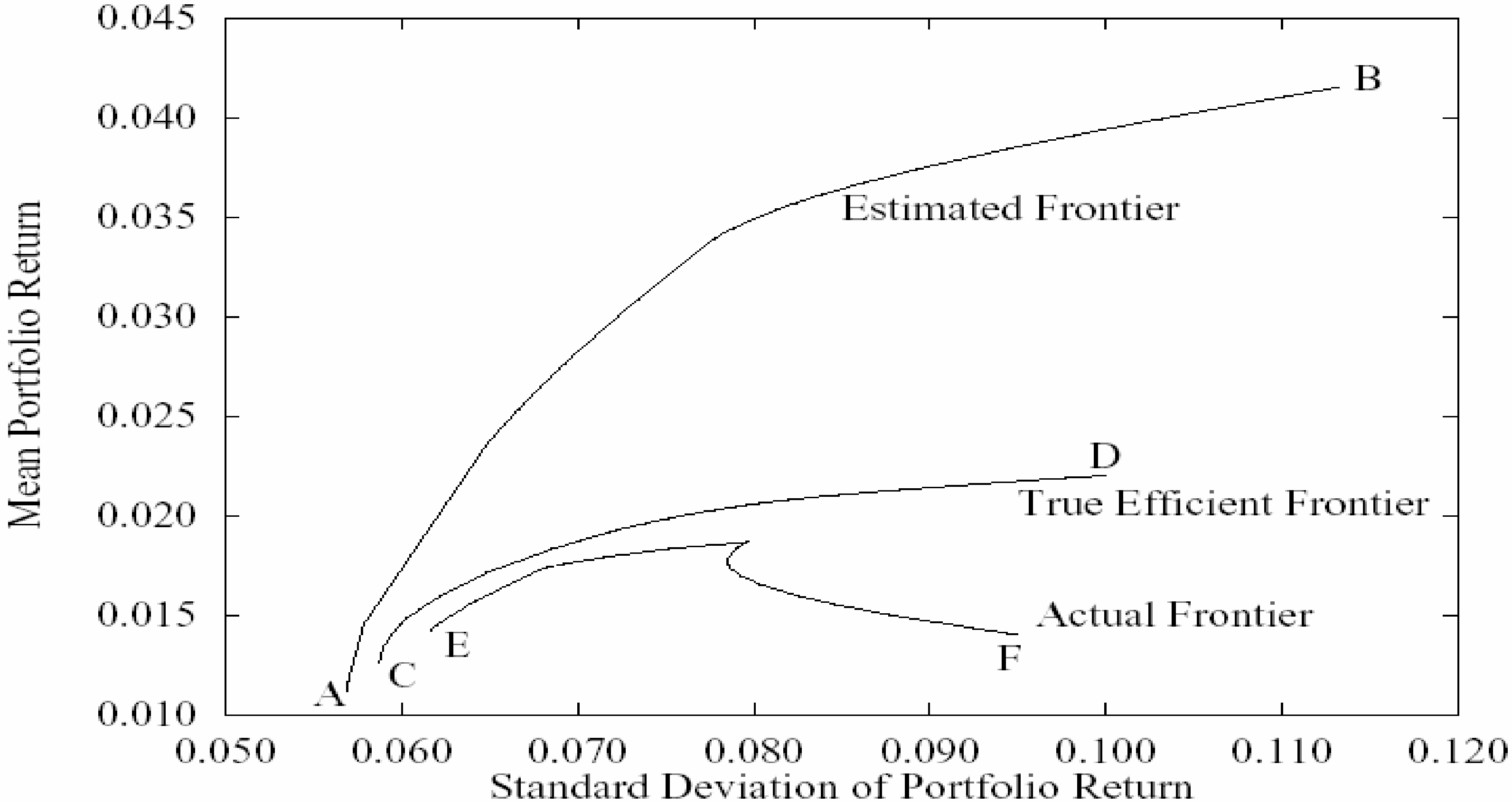
Estimating the efficient frontier

- As we vary γ so we estimate the portfolio along the frontier.

γ	Minimum Variance $=\chi/\beta = 0.16$		Tangency Portfolio $=(1+\alpha)/\chi = 23.32$		46.65	
Portfolio Weights		Std. Err		Std. Err		Std. Err
US	0.269	(0.06)	-0.263	(0.68)	-0.800	(1.37)
Canada	0.127	(0.05)	1.137	(0.65)	2.155	(1.31)
UK	0.645	(0.06)	0.656	(0.75)	0.667	(1.50)
Euro Area	-0.341	(0.06)	-0.010	(0.70)	0.324	(1.40)
Japan	0.180	(0.03)	-0.150	(0.36)	-0.482	(0.72)
Asia Pacific	0.121	(0.04)	-0.370	(0.50)	-0.864	(0.99)
Average Return, μ_W	8.49		17.73		27.03	
Volatility, σ_W	13.97		20.18		32.41	
Zero-Beta Return, r_Z	$-\infty$		0.00		4.26	
Portfolio Sharpe Ratio	*N/A*		0.88		0.70	

Quantifies the observations of Broadie (1993)

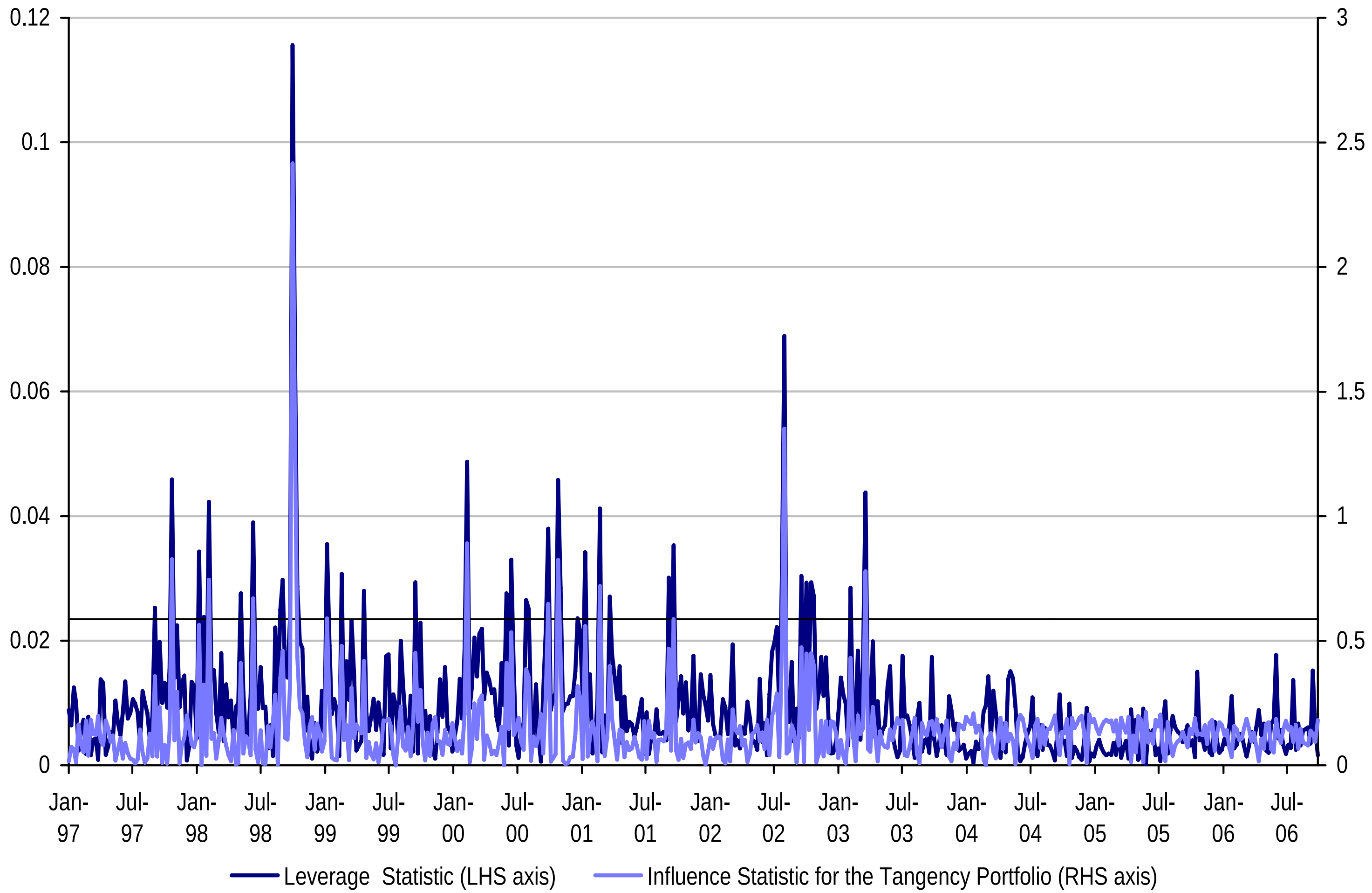
- Assume a normal distribution for 5 assets. Generate 24 months of return data.
  - True Efficient Frontier – the assumed frontier
  - Estimated Frontier – the estimated frontier
  - Actual Frontier – The achieved performance from the estimated portfolios



Source: reproduced with permission from Mark Broadie (1993), Annals of Operations Research

Leverage Statistics enables the detection of outliers

- Leverage and Influence statistic suggest September 1998 has undue impact. Dropping this **one** observation increases the underweight position on Asia by 20% and reduce the UK by 10%.



Testing Portfolio Efficiency

- For a given γ, we can test the efficiency of a portfolio *P* by an F-test of whether the estimated portfolio weights $w=w_E$ are equal to the weights of portfolio *P*.

$$Z(\gamma)=\left(\frac{T-N-1}{N-1}\right)\left(\frac{SSR\left(w=w_P\right)-SSR\left(w_E\right)}{SSR\left(w_E\right)}\right)\sim F_{N-1,T-N-1}$$

$$Z(\gamma)\approx\frac{\left[\frac{\left(\mu_E-r_Z\right)}{\sigma_E}\right]^2-\left[\frac{\left(\mu_P-r_Z\right)}{\sigma_P}\right]^2}{1+\left[\frac{\left(\mu_P-r_Z\right)}{\sigma_P}\right]^2}$$

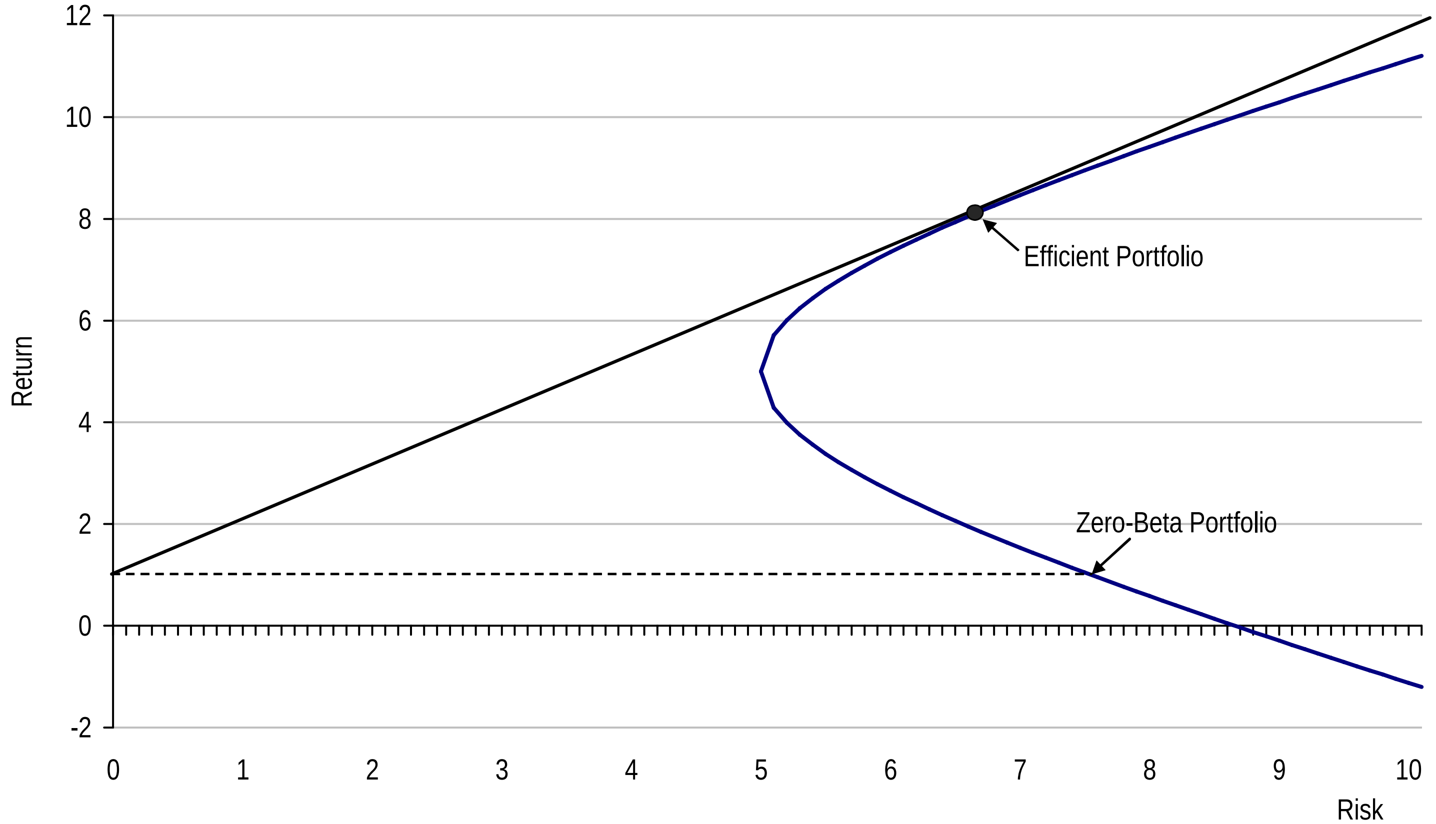
- However γ is unknown. We therefore propose as a statistic the minimum over all γ

$$Z=\underset{\gamma}{Min}\ Z(\gamma)$$

Shanken's Test of Portfolio Efficiency

- If a portfolio P is on the efficient frontier then there exists a zero-beta return, r_Z such that for any asset i

$$E(r_i) - r_Z = \beta_i \left(E(r_P) - r_Z \right)$$



Shanken's Test of Portfolio Efficiency

- Hence if a portfolio *P* is efficient frontier, there exists a zero-beta return, r_Z such that for *all* assets the constant α_i in the regression

$$r_{it} - r_Z = \alpha_i + \beta_i\left(r_{Pt} - r_Z\right) + \varepsilon_{it}$$

is equal to zero.

- Hence Shanken proposes that the efficiency of portfolio *P* can be tested by an F-test of the null hypothesis that the constants α_i are all zero. This test statistic is

$$W(r_z) = \left(\frac{T-N}{N-1}\right)\left(\frac{\alpha'\Sigma^{-1}\alpha}{1+S_P^2}\right) \sim F_{N-1,T-N}$$

where $\quad S_P = \left(\frac{(\mu_P - r_Z)}{\sigma_P}\right)$ is the Sharpe ratio of P

where Σ is the covariance matrix of the residuals ε, *T* is the number of periods and *N* the number of assets.

Shanken's Test of Portfolio Efficiency

- This supposes that the zero beta return, $r_{Z,}$ is known.
- Shanken therefore proposes taking the minimum of this statistic overall returns $r_{Z,}$

$$W = \underset{r_Z}{Min}\, W(r_z)$$

He shows that this minimum exists and gives a strict test of efficiency – in the sense that if portfolio *P* fails this test, then we can reject the hypothesis that *P* is efficient.

- Note: Finding the minimum amounts to solving a quadratic equation. We shall return to this later.

Theorem

- After a great deal of algebra, it is possible to show that

$$W(r_z) = Z(\gamma)$$

when

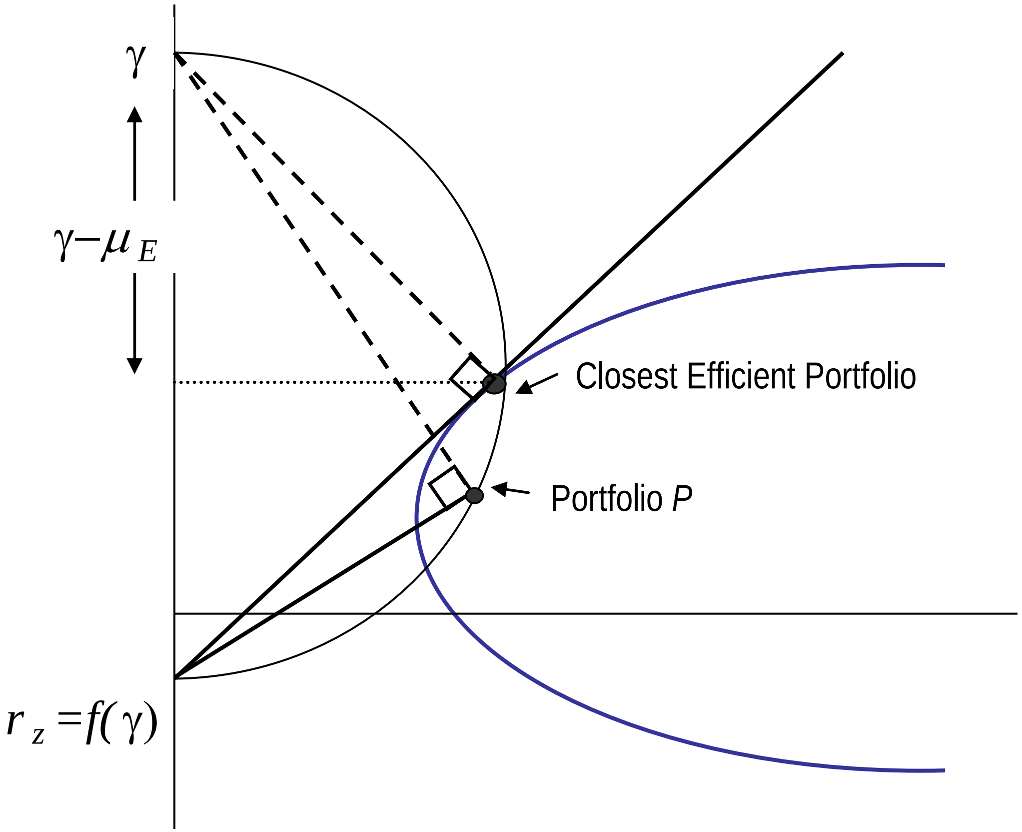
$$r_Z = \frac{(1+\alpha-\chi\gamma)}{(\chi-\beta\gamma)} \quad \text{where}$$

$$\alpha = \mu' V^{-1}\mu \qquad \beta = e_N{}' V^{-1} e_N \qquad \chi = \mu' V^{-1} e_N$$

- Hence *W* = *Z.* Further we can give this minimum a geometric interpretation in the next slide

Geometric Interpretation

- The Shanken's quadratic condition for the minimum can be rephrased as requiring that the line $r_z - P$ be perpendicular to the line γ-P.
- As we know that angles subtended on the circumference of a circle are 90^0, the circle with diameter γ- r_z connects all 'close' portfolios.



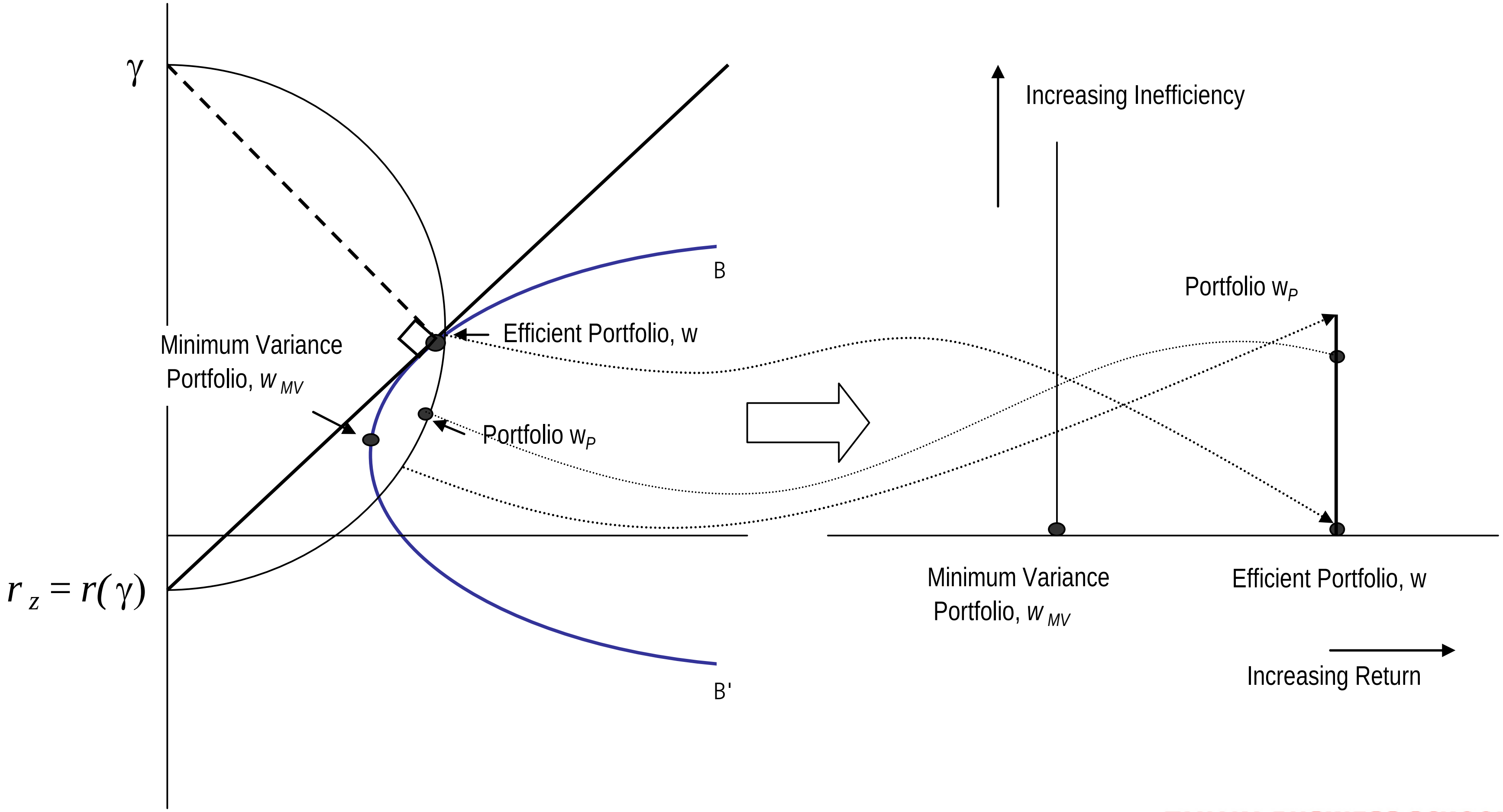
Testing the efficiency of investing only in the US

- We can test whether a portfolio invested entirely in the US is efficient, i.e. a null of $w=[1\ 0\ 0\ 0\ 0\ 0]$, at different point on the frontier

	US Portfolio			Japanese Portfolio
γ	$=\chi/\beta$ = 0.16	$=(1+\alpha)/\chi$ =23.32	92.83	0.22
Statistic $Z(\gamma)$	0.55	0.0096	0.0075	1.63
No. of Restrictions, P	5	5	5	5
$(T\text{-}N\text{-}1)/P\ Z(\gamma)$	57	0.99	0.77	167.9
p-value	0.00	0.42	0.57	0.00
Zero-Beta Return, r_Z	$-\infty$	0	6.37	-3589
Sharpe Ratio of w_P	N/A	0.55	0.19	N/A
Sharpe Ratio of w_E	N/A	0.88	0.65	N/A

The Z- distance induces a geometry

- We map the risk-return space into a return-efficiency space.
- Portfolios are different because they are more or less efficient or offer a different return.



Bayesian Extensions

- The approach can be extended very easily by extending the regression

$$\gamma e_{ext} = R_{ext} w + \varepsilon$$

where

$$e_{ext} = \begin{bmatrix} e_T \\ 1 \\ 0 \end{bmatrix} \qquad R_{ext} = \begin{bmatrix} R \\ r^{forecast} \\ V_{Prior}^{1/2} \end{bmatrix} \quad \mathrm{Var}(\varepsilon) = \sigma^2 \begin{bmatrix} I & 0 & 0 \\ 0 & t_0^{-1} & 0 \\ 0 & 0 & t_0^{-1} \end{bmatrix}$$

- The regressors are extended by the forecast returns and the prior covariance matrix. The parameter t_0 is a measure of confidence in the priors in data units.

Bayesian Extensions 2

- The risk matrix is now a weighted average of the prior and the sample risk matrix

$$V_{ext} \approx \frac{T}{T+t_0} V + \frac{t_0}{T+t_0} t_0 V_{Prior}$$

and the posterior estimates of the returns is

$$\mu_{ext} = \frac{T}{T+t_0} \mu + \frac{t_0}{T+t_0} r^{forecast}$$

- Further we could find the best portfolio *w* subject to a set of constraints on the holdings, e.g. no-short or industry neutral.

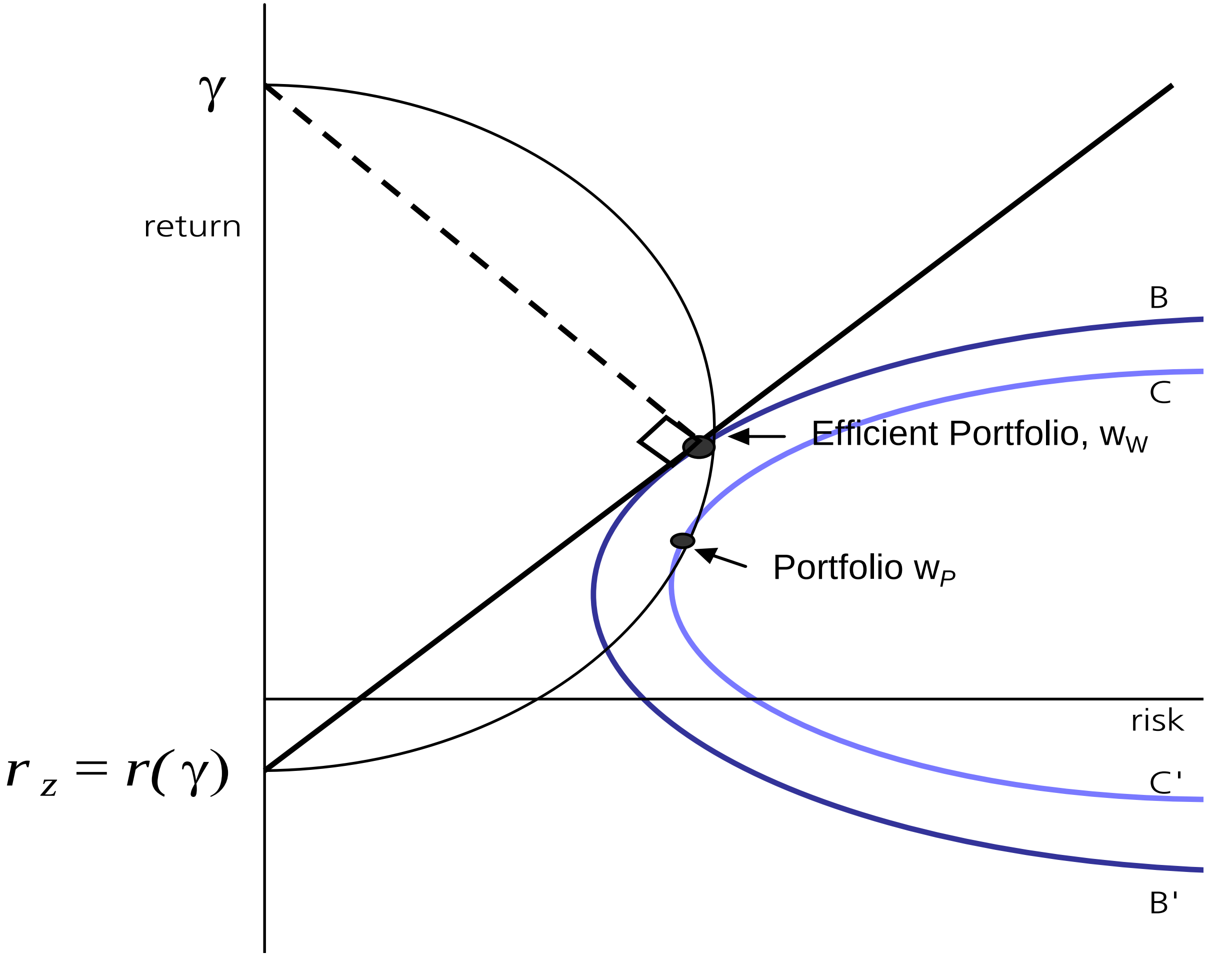
Inclusion of priors shrink the portfolios back

- We assume a single factor model and the implied equilibrium returns as our prior. As we increase t_0 so the shrinkage increases and std errors improve.

	$t_0 = 520$				$t_0 = 2080$			
γ	Minimum Variance $=\chi/\beta = 0.17$		Tangency Portfolio $=(1+\alpha)/\chi = 27.29$		Minimum Variance $=\chi/\beta = 0.17$		Tangency Portfolio $=(1+\alpha)/\chi = 29.99$	
Portfolio Weights		Std. Err		Std. Err		Std. Err		Std. Err
US	0.21	(0.03)	0.34	(0.43)	0.18	(0.02)	0.46	(0.26)
Canada	0.15	(0.03)	0.41	(0.42)	0.16	(0.02)	0.17	(0.26)
UK	0.26	(0.03)	0.24	(0.43)	0.19	(0.02)	0.16	(0.26)
Euro Area	0.05	(0.03)	0.20	(0.42)	0.13	(0.02)	0.18	(0.26)
Japan	0.18	(0.03)	-0.07	(0.33)	0.17	(0.02)	0.03	(0.24)
Asia Pacific	0.15	(0.03)	-0.12	(0.39)	0.16	(0.02)	0.00	(0.25)
Average Return, μ_W	9.10		11.21		9.04		10.03	
Volatility, σ_W	15.84		17.58		16.41		17.29	
Zero-Beta Return, r_Z	$-\infty$		0.00		$-\infty$		0.00	
Portfolio Sharpe Ratio	*N/A*		0.64		*N/A*		0.58	

We can estimate the cost of an investment constraint

- The constrained frontier lies within the efficient frontier.
- The *Z*-distance is now between a point of this frontier and its nearest neighbour on the efficient frontier.



Testing for the cost of investment constraints

- We look at two constraints. Investing only in EAFE markets and the imposing a no-short constraint.

γ	EAFE Portfolio			No Short Constraint		
	$=\chi/\beta = 0.16$	$=(1+\alpha)/\chi = 23.32$	100	$=\chi/\beta = 0.16$	$=(1+\alpha)/\chi = 23.32$	12.06
Statistic $Z(\gamma)$	0.103	0.0066	0.0047	0.066	0.0024	0.0011
No. of Restrictions, P	2	2	2	1	4	3
$(T-N-1)/P\ Z(\gamma)$	26.68	1.70	1.20	6.79	0.25	0.1143
p-value	0.00	0.18	0.30	0.01	0.91	0.9517
Zero-Beta Return, r_Z	$-\infty$	0.00	8.47	$-\infty$	0.00	-8.30
Sharpe of Constrained Portfolio	N/A	0.66	0.39	N/A	0.8.0	1.31
Sharpe Ratio of Efficient Portfolio	N/A	0.88	0.63	N/A	0.88	1.34

Conclusions

- We have rephrased our original questions into the regression framework.
- Is there sufficient new information to justify a rebalancing of my portfolio?
  - This can be rephrased as is my current portfolio, statistically different from an optimal portfolio designed using all my current information?
- Are the tilts or positions in my portfolio significantly different from the consensus or efficient set of portfolios?
  - This could be rephrased as simply whether my portfolio is significantly different from the efficient set.
  - Or more informatively, what is the confidence I need to have in my forecasts to justify my current positions?

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