Imperial College London

TANAKA BUSINESS SCHOOL

Dynamic Portfolio Optimisation

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What new questions does dynamic optimisation address?

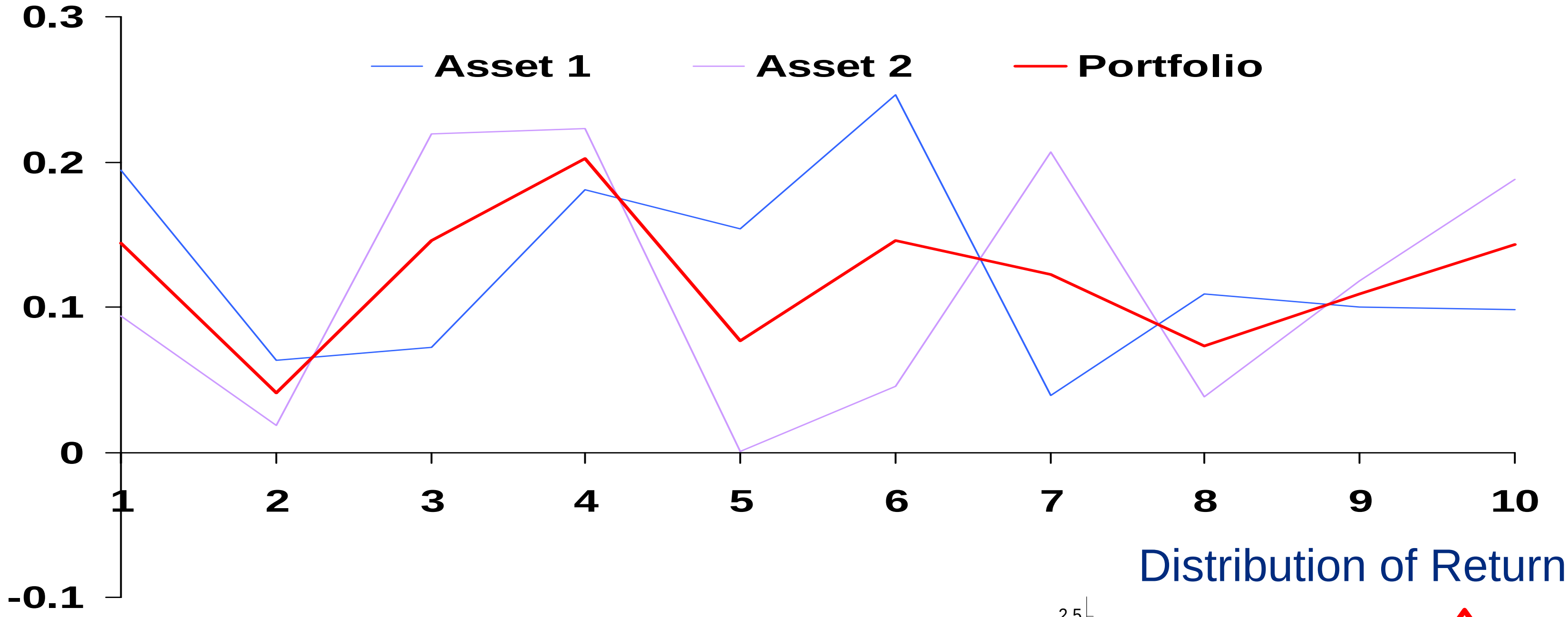
- Assume a manager has two strategies:
    1. The first strategy has a high return but decays quickly.
    2. The second has a lower expected return but decays more slowly.

  How should the manager use this information in constructing his portfolio?

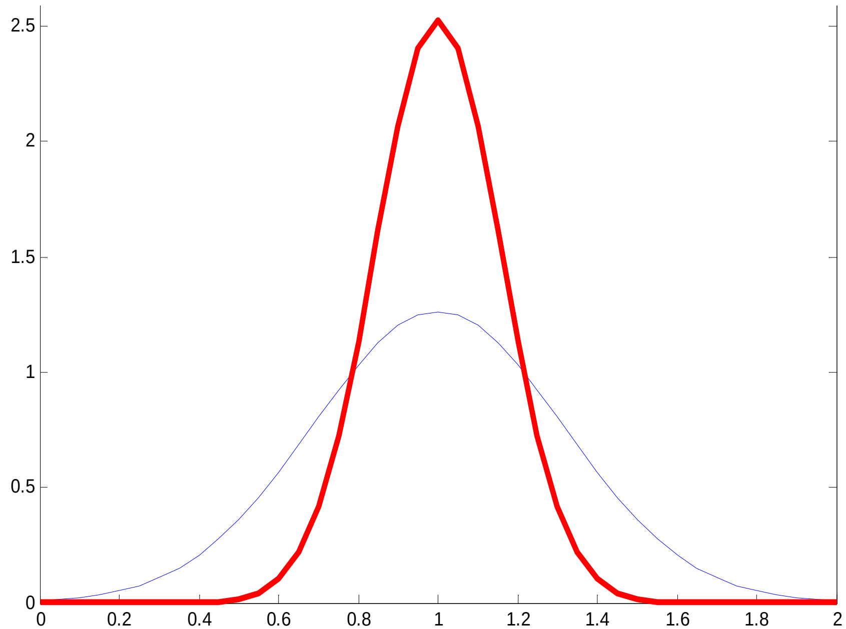
- Now take into account transaction costs and problem becomes interesting!

- Multiperiod optimisation introduces hedging motives for holding some assets. These asset hedge against future falls in expected returns in the strategies (or the future investment opportunity set).

Diversification of within period risk



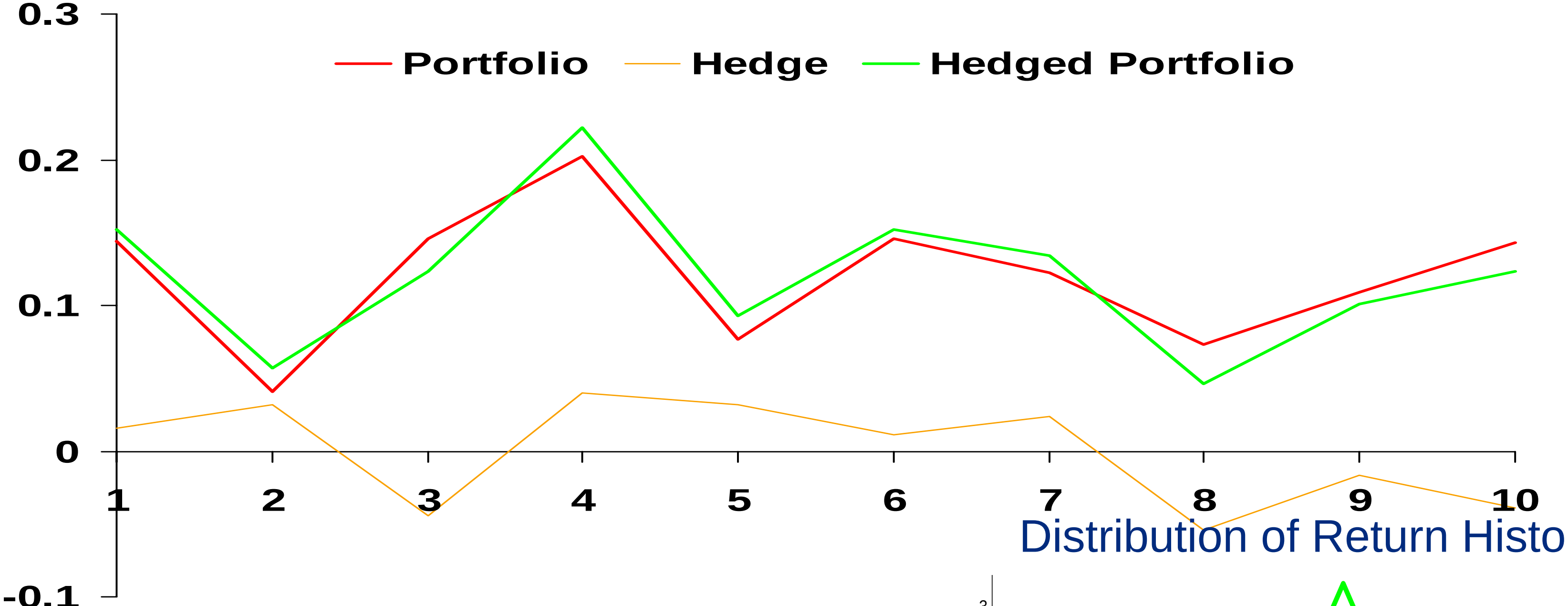
Distribution of Return Histories



- Construct a portfolio from two assets that are negatively correlated in each period. Within period variance is less than the variance of each asset.

- The distribution of the return histories over *n* periods has mean and variance equal to *n* times the within period mean and variance.

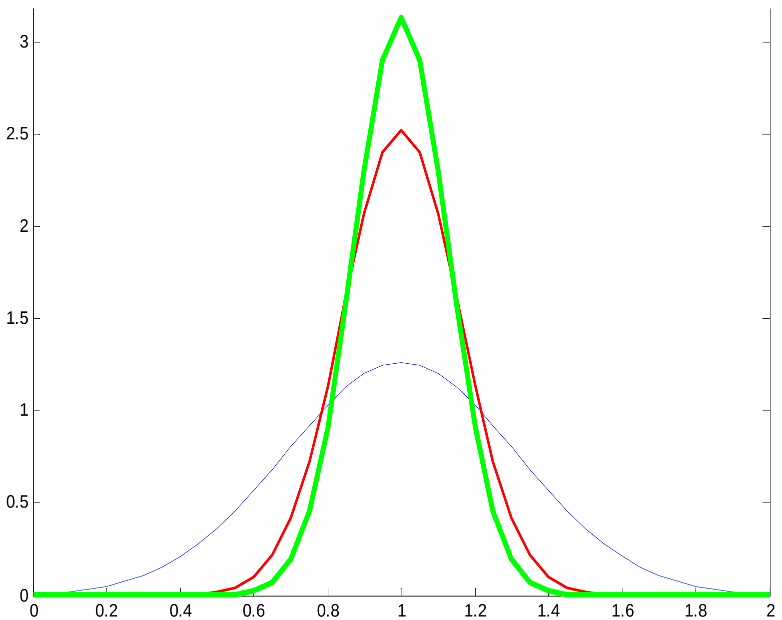
Time diversification of risk



- Construct a hedged portfolio from earlier portfolio and a hedge where hedge is negatively correlated to portfolio **across** periods. Within period variance can be greater than the variance of unhedged portfolio.

- **But** the distribution of the return histories over *n* periods has mean and variance less than *n* times the within period mean and variance.

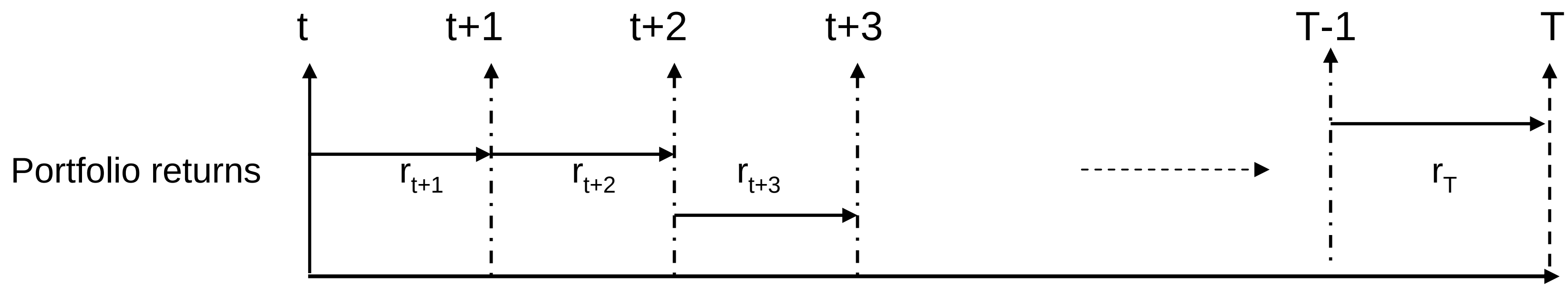
Distribution of Return Histories



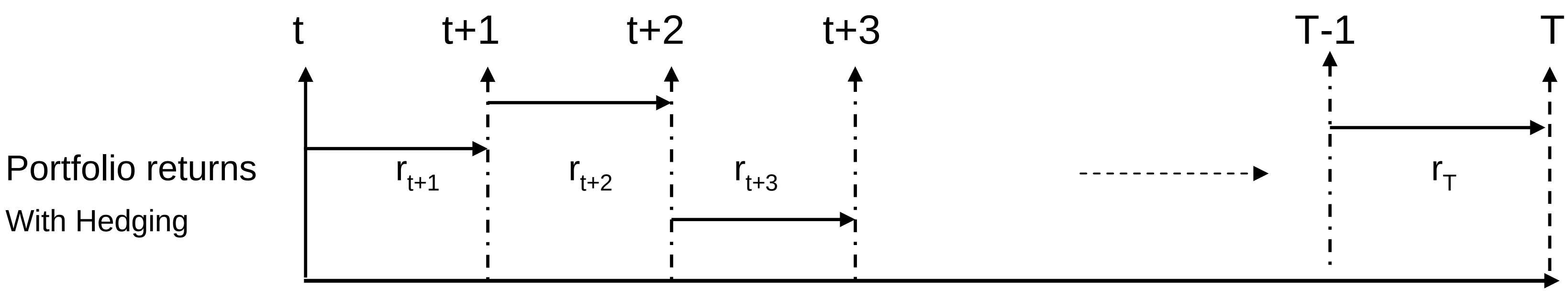
An example of a Hedging Portfolio

- Assume returns, *r*, in any period are distributed
  - Risk Premiums are constant but the risk free rate varies over time $r_t \sim N(r_t^{free} + \mu, V)$

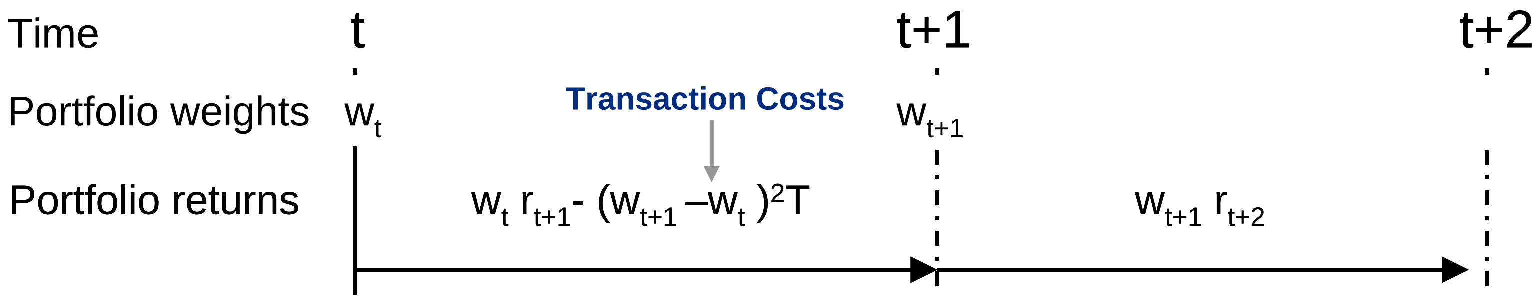
Then returns to optimal portfolio will vary with the risk free rate.



- Now assume there is a 'hedging' portfolio whose returns are negatively correlated to changes in the risk free rate. If we hold some of this portfolio we can smooth our performance over time.



Return Predictability and Transaction Costs can induce correlation across periods



- Let $r^e_{t+i} = E_{t+i-1}\left(r_{t+i}\right)$ denote expected returns at the beginning of the period. Let the portfolio construction rule be $w_t = K r^e_{t+1}$ and define the system equations as follows

$$r^e_{t+2} = A r^e_{t+1} + \xi_{t+1}$$

$$r_{t+i} = r^e_{t+i} + \varepsilon_{t+1} \quad \text{for } i = 1, 2$$

where ξ, ε are i.i.d. random variables.

- Note the following

  1. Returns are just equal to expected returns plus an innovation.
  2. Expected returns are time-varying if $\xi \neq 0$ and predictable if $A \neq 0$.
  3. The portfolio rule can be any function of expected returns – this is just the simplest.
  4. Transaction cost can be any function of the change in portfolio weights.

Return Predictability and Transaction Costs can induce correlation across periods

- Recall the system equations

$$r^e_{t+2} = Ar^e_{t+1} + \xi_{t+1}$$

$$r_{t+1} = r^e_{t+1} + \varepsilon_{t+1}$$

$$r_{t+2} = r^e_{t+2} + \varepsilon_{t+2} = Ar^e_{t+1} + \xi_{t+1} + \varepsilon_{t+2}$$

- The returns to the portfolio in periods 1 and 2 are:

$$\text{Return period 1} := w_t r_{t+1} - T\left(w_{t+1} - w_t\right)^2 = Kr^e_{t+1}\left(r^e_{t+1} + \varepsilon_{t+1}\right) - TK\left((A-1)r^e_{t+1} + \xi_{t+1}\right)^2$$

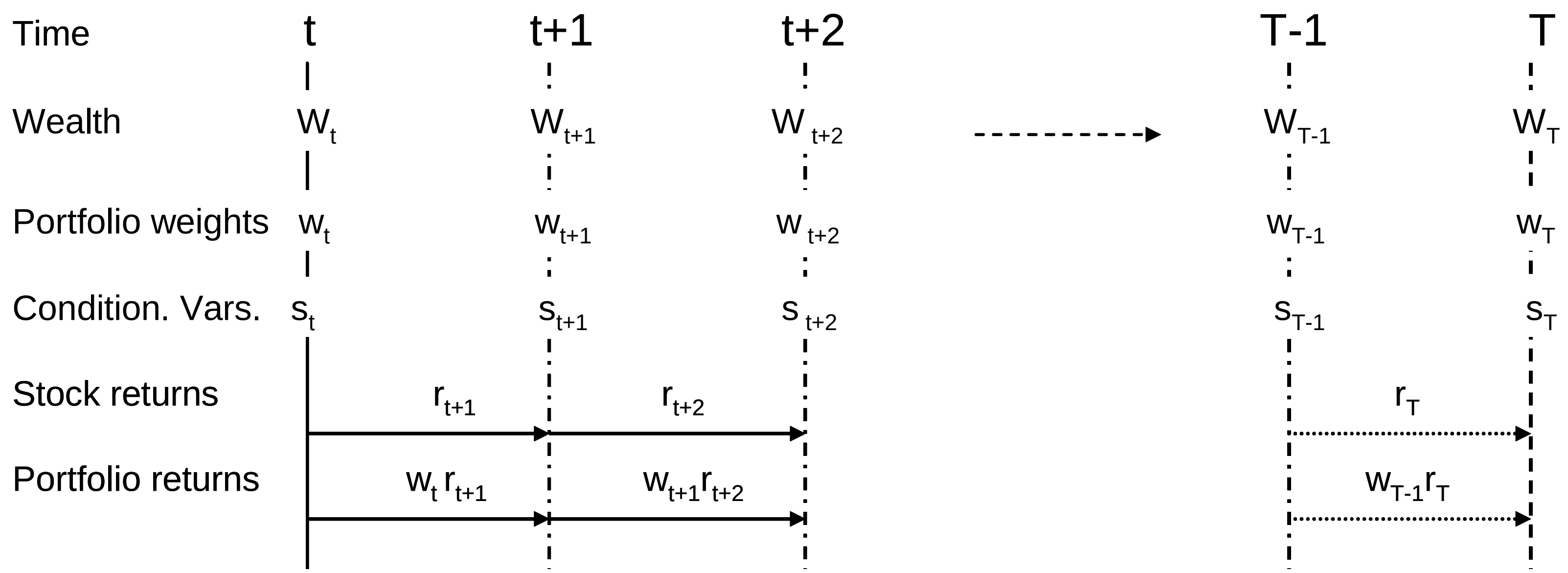
$$\text{Return period 2} := w_{t+1} r_{t+2} = K\left(Ar^e_{t+1} + \xi_{t+1}\right)\left(Ar^e_{t+1} + \xi_{t+1} + \varepsilon_{t+2}\right)$$

- And so returns are correlated across periods if:

$\mathrm{E}\left(\varepsilon_{t+1}\xi_{t+1}\right) \neq 0$ These are the classical Merton hedging demands

$T \neq 0$ and $\xi_{t+1} \neq 0$ Time-varying expected returns and non-zero transaction costs

The Dynamic Portfolio Construction Problem



- The problem is to choose portfolios, w_t, to maximise the performance measure

$$\frac{1}{1-\gamma}\log\left[E_t\left(W_T^{1-\gamma}\right)\right]$$

- This is just mean-variance extended to the multi-period problem. For if

$$W_T = W_t e^{r} \approx W_t(1+r) \quad \text{then}$$

$$\frac{1}{1-\gamma}\log E_t\left(W_T^{1-\gamma}\right) = \log W_t + E(r) + \frac{1-\gamma}{2}Var(r)$$

A Tractable Optimisation Criterion

- We use Campbell & Viceria (2003) log-linear approximation. Let *$Var(r_t)=\Sigma$ and $\sigma^2=diag(\Sigma)$*

$$W_T = W_t\left(w_t' e^{r_{t+1}}\right)\left(w_{t+1}' e^{r_{t+2}}\right)\ldots.$$

$$\approx W_t\left(e^{w_t' r_{t+1}+\frac{1}{2}w_t'\left(\sigma^2-\Sigma w_t\right)}\right)\left(e^{w_{t+1}' r_{t+2}+\frac{1}{2}w_{t+1}'\left(\sigma^2-\Sigma w_{t+1}\right)}\right)\ldots. = W_t\left(e^{\sum_{i=0}^{T-1} w_{t+i}' r_{t+1+i}+\frac{1}{2}w_{t+i}'\left(\sigma^2-\Sigma w_{t+i}\right)}\right)$$

- And substitute this into our performance measure

$$\frac{1}{1-\gamma}\log\left[E_t\left(W_T^{\,1-\gamma}\right)\right] = \log W_t + \frac{1}{1-\gamma}\log\left[E_t\left(e^{(1-\gamma)\sum_{i=0}^{T-1} w_{t+i}' r_{t+1+i}+\frac{1}{2}w_{t+i}'\left(\sigma^2-\Sigma w_{t+i}\right)}\right)\right]$$

- This is an exponential of a quadratic cost or risk-sensitive control problem studied in

Peter Whittle (1990). *Risk Sensitive Optimal Control*, Wiley & Sons.

Iglesias, P.A. and Glover, K. (1991). ‘State-space approach to discrete-time *$H\infty$* Control’, *International Journal of Control,* Vol 54(5), pp1031-1073.

Why is it tractable?

- The tractability follows directly from *Lemma 6.1.2*, Whittle (1990). If $\Xi(w,r,\varepsilon)$ is a quadratic function in all 3 arguments and positive definite in ε then

$$\min_{w} \log E_{\varepsilon} \left(e^{-\frac{1}{2}\Xi(r,w,\varepsilon)} \right) \propto -\frac{1}{2} \max_{w} \min_{\varepsilon} \Xi(r,w,\varepsilon)$$

Hence the problem reduces to finding the best portfolio *w* in the face of the worst case disturbance ε.

- Others have studied similar problems in the finance literature but have not used the explicit solutions developed in the control literature.

Campbell, John and Viceria (2003). 'A multivariate model of asset allocation', Review of Financial Studies.

Bielecki, Stanley R. and Pliska (2005). 'Risk sensitive asset management with t transaction costs', Finance and Stochastics

Litterman, Robert (2006). 'Multi –Period Portfolio Optimisation. Presentation to Risk Conference.

Strategic Asset Allocation – Barberis (2000)

- An allocator chooses between investing in a risk free asset with return $r_{f,}$= 0.0036 and risky equity asset with excess returns, $r_{t,}$.

- The dividend price ratio *(d/p)* forecasts future equity returns. The following system was estimated on monthly US data between 1986-1995

$$\begin{bmatrix} r_{t+1} \\ (d/p)_{t+1} \end{bmatrix} = \begin{bmatrix} -0.0303 \\ 0.0013 \end{bmatrix} + \begin{bmatrix} 1.0919 \\ 0.9577 \end{bmatrix} (d/p)_t + \begin{bmatrix} \varepsilon_{t+1} \\ \xi_{t+1} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_t \\ \xi_t \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.0019 & \boldsymbol{-0.9323} \\ & 2.6E-6 \end{bmatrix} \right)$$

Correlation coefficient (← -0.9323)

- A negative shock, ξ, to the dividend ratio presages lower future equity returns.

- However ε, is negatively correlated with such a shock => Equity is a hedge against a fall in future returns.

- The problem is to choose the portfolio allocation to equity, w_t

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Strategic Asset Allocation – Barberis (2000)

- We note $r^e_{t+1} = 1.046(d/p)_t - 0.0303$ and rewrite the system in the general form

$$\begin{bmatrix} r^e_{t+2} \\ (d/p)_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1.046 \\ 0 & 0.958 \end{bmatrix} \begin{bmatrix} r^e_{t+1} \\ (d/p)_t \end{bmatrix} + \begin{bmatrix} 1.0919 \\ 1 \end{bmatrix} \xi_{t+1} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} w_t + \begin{bmatrix} -0.029 \\ 0.0013 \end{bmatrix}$$

$$z_t = \begin{bmatrix} r_{t+1} \\ w_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r^e_{t+1} \\ (d/p)_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \varepsilon_{t+1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_t$$

$$y_t = \begin{bmatrix} r^e_{t+1} \\ (d/p)_t \end{bmatrix}$$

Negatively Correlated

- The within period cost is a quadratic function of outputs z_t

$$w_t' r_{t+1} + \frac{1}{2} w_t' \left(\sigma^2 - \Sigma w_t\right) = \begin{bmatrix} r_{t+1} \\ w_t \end{bmatrix}' \begin{bmatrix} 0 & 1/2 \\ 1/2 & -0.0009 \end{bmatrix} \begin{bmatrix} r_{t+1} \\ w_t \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0.0151 \end{bmatrix}' \begin{bmatrix} r_{t+1} \\ w_t \end{bmatrix}$$

- And the portfolio allocation problem amounts to find the best control from outputs *y*

$$w_t = K y_t + W_0$$

Solution to the asset allocation problem

- The value function is quadratic in the dividend ratio $(d/p)_t$

$$\frac{1}{1-\gamma}\log\left[E_t\left(W_T^{1-\gamma}\right)\right] = \Pi_t\left(\frac{d}{p}\right)_t^2 - 2\Phi_t\left(\frac{d}{p}\right)_t + \phi_t$$

Both Π_t and Φ_t are found by backward iteration, with initial values $\Pi_T = \Phi_T = 0$, where

$$\Pi_t = \frac{1.45E6(1-\gamma)\Pi_{t+1} + 4.61E8}{(1-\gamma)^2\Pi_{t+1} + 1.469E6(1-\gamma)}$$

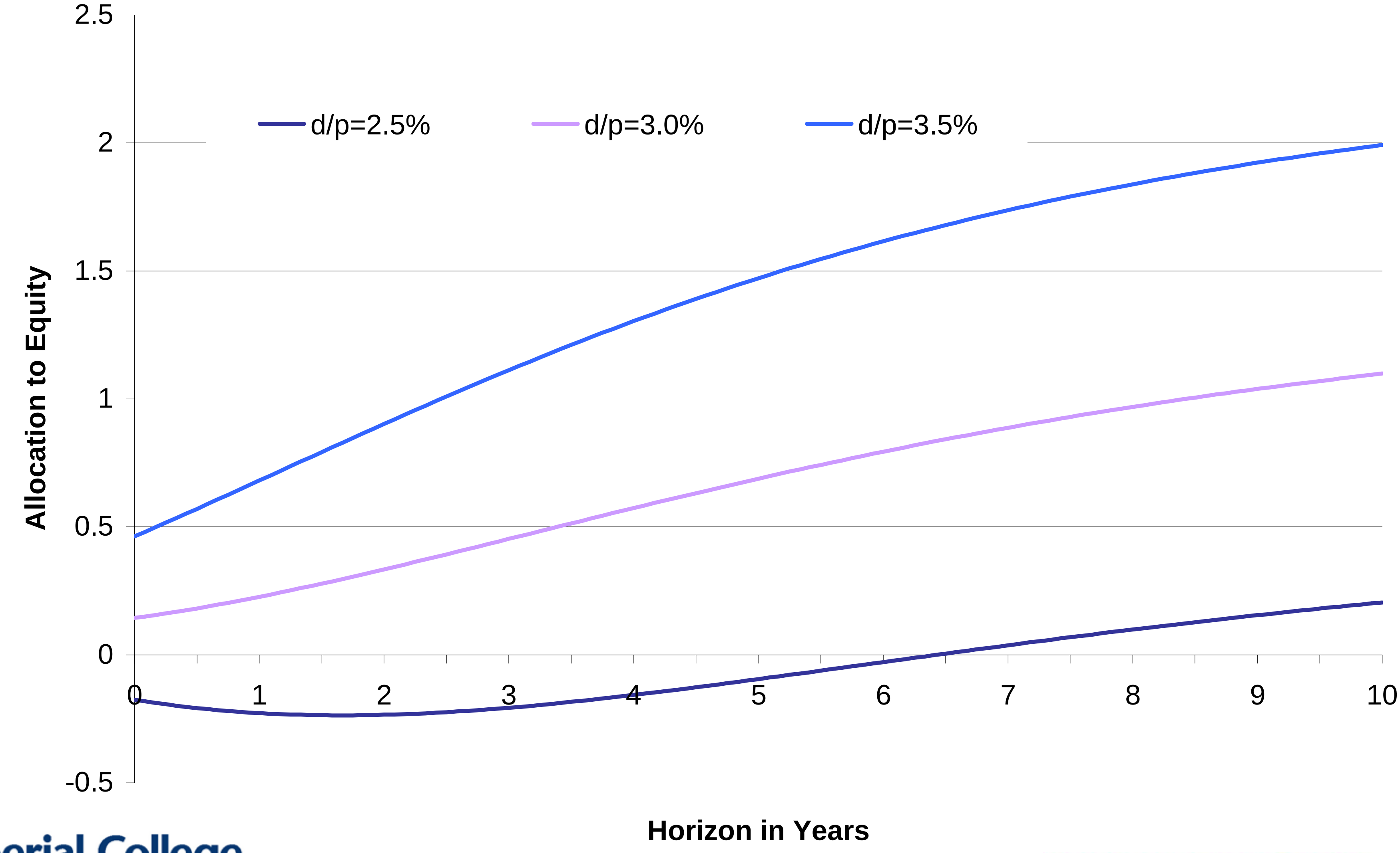
$$\Phi_t = \frac{1.46E6(1-\gamma)\Phi_{t+1} - 364.2(1-\gamma)\Pi_{t+1} + 1.27E7}{(1-\gamma)^2\Pi_{t+1} + 1.469E6(1-\gamma)}$$

- The optimal allocation to equity is

$$w_t = \frac{\left(8.44E8\left(\frac{d}{p}\right)_t - 2.34E7\right) + (1-\gamma)\Pi_{t+1}\left(1.01E5\left(\frac{d}{p}\right)_t + 9.89\right) - 1.01E5(1-\gamma)\Phi_{t+1}}{(1-\gamma)^2\Pi_{t+1} + 1.469E6(1-\gamma)}$$

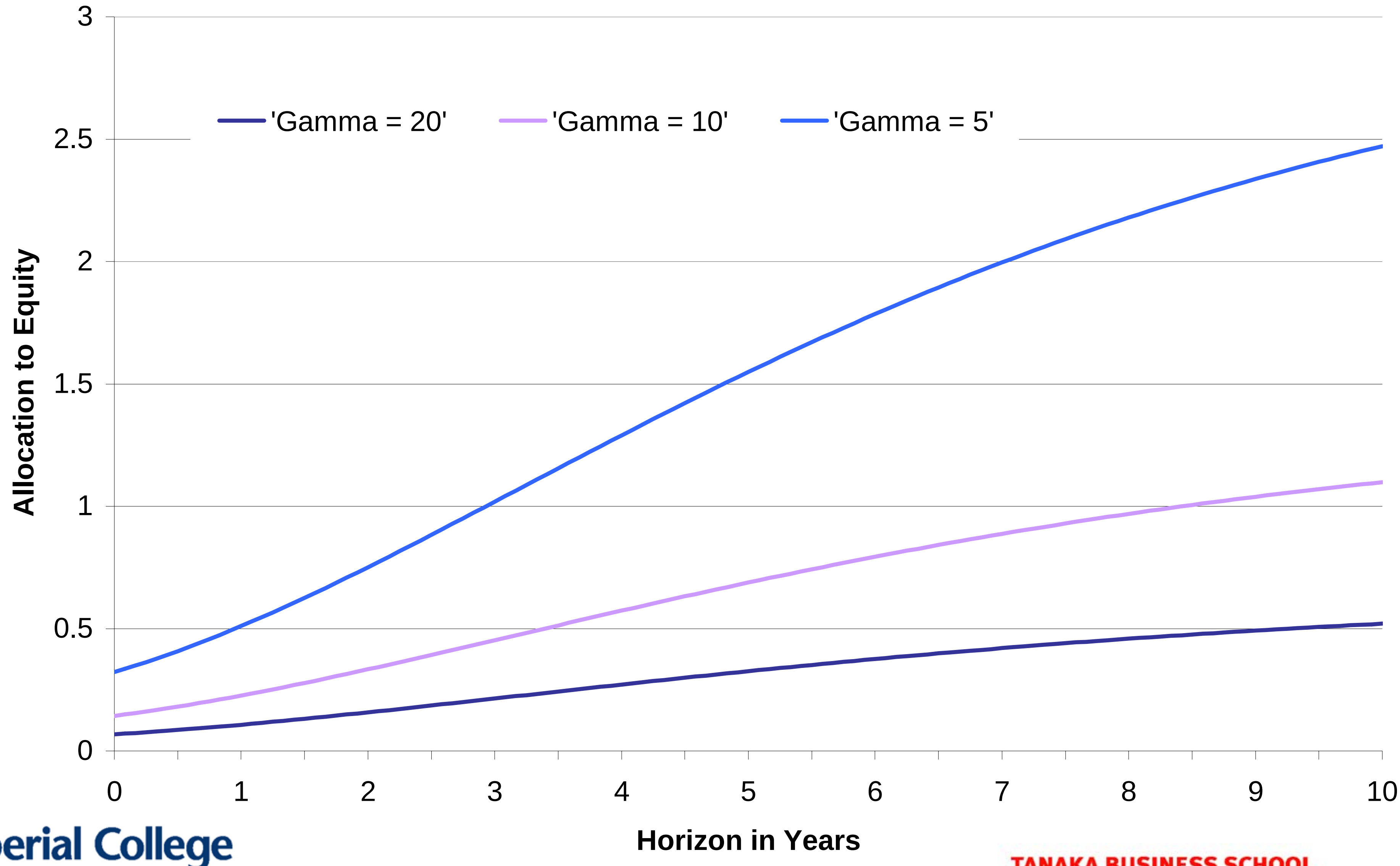
The longer the horizon, the greater allocation to equity

- The allocation to equity as a function of both the dividend / price ratio and the investment horizon given $\gamma = 10$.



The more risk averse, the lower the allocation to equity

- The allocation to equity as a function of the coefficient of risk aversion and the investment horizon for a dividend/price ratio of 3%



Performance Comparison

- Comparison of three portfolio construction techniques over a 10 year horizon
  - We compare the average annualised log of the excess returns over the 10 years
  - The risk aversion parameter of single period mean variance rule is chosen so as to have the same sample variance as the dynamic optimisation portfolio.

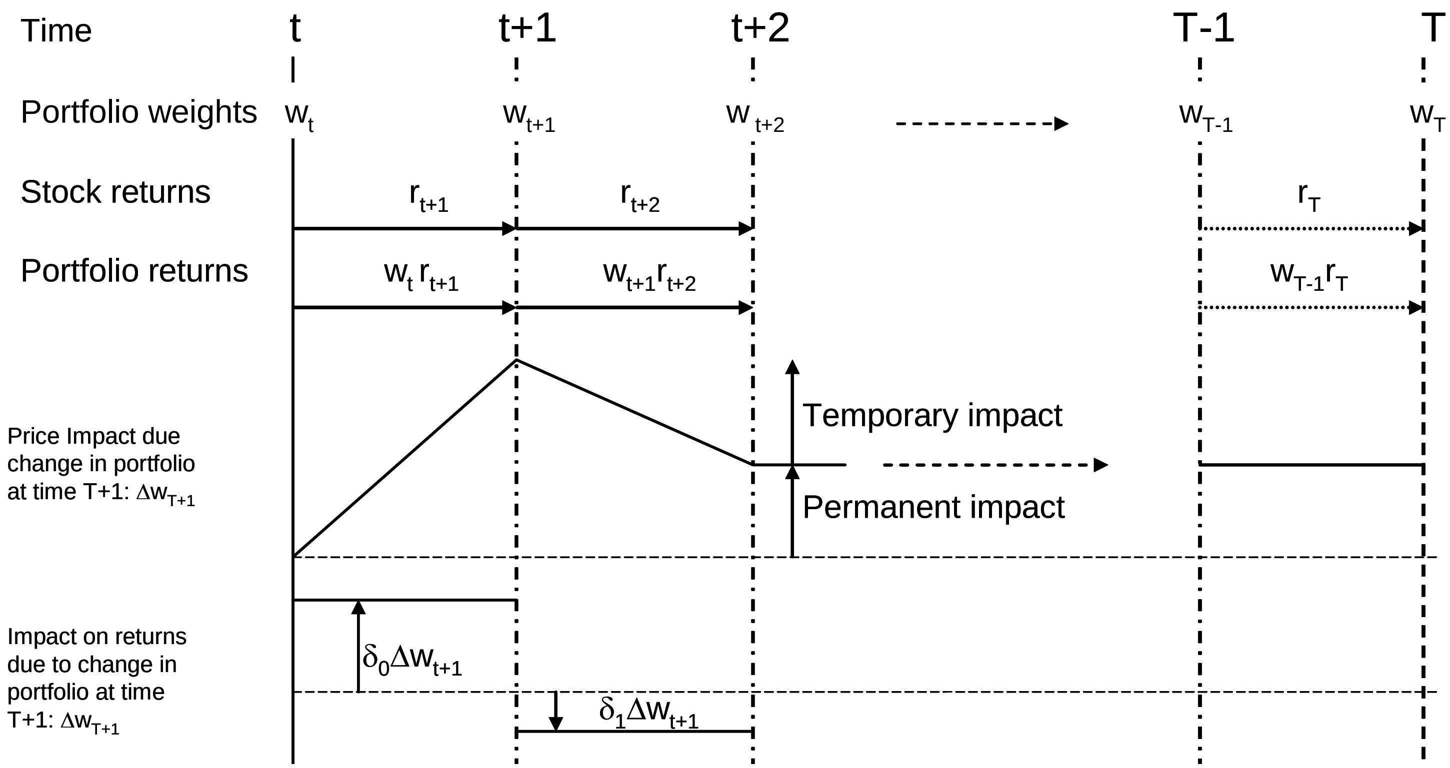
	100% Equity Allocation	Single Period Mean Variance	Dynamic Optimisation
		$\gamma = 5$	
Mean Excess Return	0.024	0.128	0.146
Std. Dev.	0.073	0.155	0.155
Sharpe	0.330	0.826	0.947
		$\gamma = 10$	
Mean Excess Return	0.024	0.058	0.066
Std. Dev.	0.070	0.067	0.067
Sharpe	0.347	0.856	0.977
		$\gamma = 20$	
Mean Excess Return	0.024	0.028	0.032
Std. Dev.	0.071	0.033	0.033
Sharpe	0.338	0.861	0.950

Modelling Transaction Costs

- We model both the explicit cost as

$$\pounds_t^{Explicit} = -\Delta w_t' T \Delta w_t$$

- And the implicit cost due to the price impact of the trade



$$r_{t+1} = r_{t+1}\big|_{No\,Trade} + \delta_0 \Delta w_{t+1} - \delta_1 \Delta w_t$$

The Investment Environment

- We shall describe the forecasting model by the following set of equations

$$\begin{bmatrix} r^e_{t+2|t+1} \\ s_{t+1} \\ w_t \\ \Delta w_t \end{bmatrix} = \begin{bmatrix} A_r & A_{r,s} & 0 & 0 \\ 0 & A_s & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r^e_{t+1|t} \\ s_t \\ w_{t-1} \\ \Delta w_{t-1} \end{bmatrix} + \begin{bmatrix} B_{1,1} \\ B_{2,1} \\ 0 \\ 0 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 0 \\ I \\ I \end{bmatrix} \Delta w_t + \begin{bmatrix} m_r \\ m_s \\ 0 \\ 0 \end{bmatrix}$$

where $r^e_{t+1|t} = E_t\left(r_{t+1} \middle| I_t\right)$ and s_t are conditioning (possibly macro) variables. We shall refer to the vector $x_t = \begin{bmatrix} r^{e\prime}_{t+1|t} & s_t' & w_{t-1}' & \Delta w_{t-1}' \end{bmatrix}'$ as the states.

- It will prove useful to write the output variables are a function of the states

$$z_t = \begin{bmatrix} r_{t+1} \\ w_t \\ \Delta w_t \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r^e_{t+1|t} \\ s_t \\ w_{t-1} \\ \Delta w_{t-1} \end{bmatrix} + \begin{bmatrix} D_{11,r} \\ 0 \\ 0 \end{bmatrix} x_t + \begin{bmatrix} -d_1 \\ I \\ I \end{bmatrix} \Delta w_t$$

The Performance Criterion

- The cost function is a function of these performance variables

$$
\begin{aligned}
\Gamma_t^{Total} &= r_{t+1}' w_t - \Delta w_t' T \Delta w_t \\
&= \begin{bmatrix} r_{t+1} \\ w_t \\ \Delta w_t \end{bmatrix}' \begin{bmatrix} 0 & \tfrac{1}{2} I & 0 \\ \tfrac{1}{2} I & 0 & 0 \\ 0 & 0 & -T \end{bmatrix} \begin{bmatrix} r_{t+1} \\ w_t \\ \Delta w_t \end{bmatrix} \\
&= z_t' R z_t - 2 S z_t
\end{aligned}
$$

N.B above $\boldsymbol{S} = 0$, but in later examples this extra term will prove to be useful.

- The problem is to choose Δw_t so as to maximise the criterion

$$U_t = -\frac{2}{\theta} \log_e E_t \left(e^{-\frac{\theta}{2} \sum_{i=t+1}^{T} \Gamma_{i-1}^{Total}} \right)$$

Solution to Dynamic Portfolio Construction Problem

- This is a minor extension of the results in Whittle (1990) and Iglesias and Glover (1991). For brevity rewrite out system in terms of the *n* state vector *x*

$$x_{t+1} = Ax_t + B_1 x_{t+1} + B_2 \Delta w_t + m$$

$$z_t = C_1 x_t + D_{11} x_{t+1} + D_{12} \Delta w_t$$

where *Var(ξ)= Ω*. The solution is expressed in terms of the *nxn* matrix Π and n-vector Φ. The matrix Π is a solution to the *H∞* Riccati equation. To this end define

$$V_{t+1} = \begin{bmatrix} D_{11}' \\ D_{12}' \end{bmatrix} R \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} + \begin{bmatrix} B_1' \\ B_2' \end{bmatrix} \Pi_{t+1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} + \begin{bmatrix} (\theta\Omega)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$K_{t+1} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = V^{-1} \left(\begin{bmatrix} D_{11}' \\ D_{12}' \end{bmatrix} R C_1 + \begin{bmatrix} B_1' \\ B_2' \end{bmatrix} \Pi_{t+1} A \right)$$

$$W_{t+1} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = V^{-1} \left(\begin{bmatrix} D_{11}' \\ D_{12}' \end{bmatrix} S + \begin{bmatrix} B_1' \\ B_2' \end{bmatrix} \left(\Phi_{t+1} - \Pi_{t+1} m \right) \right)$$

Solution to Dynamic Portfolio Construction Problem (2)

- Then the matrix Π is a solution to the following $H\infty$ Riccati equation

$$\Pi_t = A'\Pi_{t+1}A + C_1'RC_1 - K_{t+1}'V_{t+1}^{-1}K_{t+1}$$

and Φ solves the linear equation

$$\Phi_t = C_1'S + A'\left(\Phi_{t+1} - \Pi_{t+1}m\right) - K_{t+1}'V_{t+1}W_{t+1}$$

- These equations can be solved by backward iteration. Set both Π and Φ to zero, substitute into the right hand side of the above equations. The next solution si given by the left hand side. Iterate until convergence.
  - The steady state can alose be found by solving an eigenvector-eigenvalue problem
- The optimal change to the portfolio weights is then calculated as

$$\Delta w_t = -K_2x_t + W_2$$

An example

- Construct a portfolio of the following 3 assets subject to being fully invested
  - Long-Short US Momentum Portfolio
  - Long-Short US Value Portfolio.
  - US Market Portfolio (fully invested constraint implies a weight of 1 on the market).

- The forecast model of expected returns was constructed as follows:

1. Estimate a VAR (vector autoregression) on the returns to the portfolios and the following conditioning variables: Short rates, Credit and Term Spreads, Industrial Production, Inflation, Dividend Yields, X-section and Market Volatility on monthly data 1984 – 2005.

2. From this VAR construct a time-series of forecast of expected returns to the portfolios.

3. From this time series of forecasts estimate another VAR describing the evolution of these forecasts.

The Forecasting Model

- The state-space evolution equations is

$$
\begin{bmatrix} r^{Mom}_{t+2|t+1} \\ r^{Val}_{t+2|t+1} \\ r^{Mkt}_{t+2|t+1} \\ s^{Term}_{t+1} \\ w^{Mom}_{t} \\ w^{Val}_{t} \\ \Delta w^{Mom}_{t} \\ \Delta w^{Val}_{t} \end{bmatrix}
=
\begin{bmatrix}
0.45 & 0.00 & -0.04 & 0 & 0 & 0 & 0 & 0 \\
0.00 & 0.21 & -0.24 & 2.91 & 0 & 0 & 0 & 0 \\
0.2 & -0.1 & 0.68 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.98 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} r^{Mom}_{t+1|t} \\ r^{Val}_{t+1|t} \\ r^{Mkt}_{t+1|t} \\ s^{Term}_{t} \\ w^{Mom}_{t-1} \\ w^{Val}_{t-1} \\ \Delta w^{Mom}_{t-1} \\ \Delta w^{Val}_{t-1} \end{bmatrix}
+
\begin{bmatrix} x^{Mom}_{t+1} \\ x^{Val}_{t+1} \\ x^{Mkt}_{t+1} \\ x^{Term}_{t+1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+
\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
\begin{bmatrix} \Delta w^{Mom}_{t} \\ \Delta w^{Val}_{t} \end{bmatrix}
+
\begin{bmatrix} .0044 \\ .0022 \\ .0025 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Where

$$
Var\begin{pmatrix} x^{Mom}_{t+1} \\ x^{Val}_{t+1} \\ x^{Mkt}_{t+1} \\ x^{Term}_{t+1} \end{pmatrix}
=
\begin{bmatrix}
\mathbf{1.10\%} & -0.08 & 0.52 & -0.09 \\
- & \mathbf{2.73\%} & -0.5 & -0.024 \\
- & - & \mathbf{2.86\%} & 0.4 \\
- & - & - & \mathbf{.05\%}
\end{bmatrix}
$$

Bold Elements are annualised volatilities, Italic elements are correlations

Note

- Low variance of expected momentum returns
- Value and the Market returns are negatively correlated

The Model Output

- The output equation are

$$\begin{bmatrix} r_{t+1}^{Mom} \\ r_{t+1}^{Val} \\ r_{t+1}^{Mkt} \\ w_t^{Mom} \\ w_t^{Val} \\ \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{t+1|t}^{Mom} \\ r_{t+1|t}^{Val} \\ r_{t+1|t}^{Mkt} \\ s_t^{Term} \\ w_{t-1}^{Mom} \\ w_{t-1}^{Val} \\ \Delta w_{t-1}^{Mom} \\ \Delta w_{t-1}^{Val} \end{bmatrix} + \begin{bmatrix} e_{t+1}^{Mom} \\ e_{t+1}^{Val} \\ e_{t+1}^{Mkt} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.005 & 0 \\ 0 & -0.005 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix}$$

Where

$$Var\begin{pmatrix} e_{t+1}^{Mom} \\ e_{t+1}^{Val} \\ e_{t+1}^{Mkt} \end{pmatrix} = \begin{bmatrix} \mathbf{13.61\%} & 0.03 & 0.02 \\ - & \mathbf{8.59\%} & -0.48 \\ - & - & \mathbf{14.58\%} \end{bmatrix}$$

Bold Elements are annualised volatilities, Italic elements are correlations

Note

- High variance of the market and momentum returns
- Realised value and the Market returns are also negatively correlated

Hedging Opportunities

- To reduce variance of end of period wealth, investors would like to hold assets that deliver high returns when future investment opportunities decline.

- Therefore hedging involves the correlations between the innovations ε and ξ

$$Cor\left(\begin{bmatrix} \xi_{t+1}^{Mom} \\ \xi_{t+1}^{Val} \\ \xi_{t+1}^{Mkt} \\ \xi_{t+1}^{Term} \end{bmatrix}, \begin{bmatrix} \varepsilon_{t+1}^{Mom} \\ \varepsilon_{t+1}^{Val} \\ \varepsilon_{t+1}^{Mkt} \end{bmatrix} \right) = \begin{bmatrix} -.05 & 0.01 & -0.01 \\ .59 & -.12 & 0.45 \\ .05 & -.53 & 0.09 \\ .19 & .05 & 0.06 \end{bmatrix}$$

- So

1. Value is a hedge against the poor future market returns – When ξ^{Mkt} is negative, ε^{Val} tends to be positive

2. Momentum is a hedge against poor returns to value – When ξ^{Val} is negative, ε^{Mom} tends to be positive

The Performance Criteria

- The cost function is a function of these output variables

$$\mathcal{L}_t^{Total} = r_{t+1}' w_t - \Delta w_t' T \Delta w_t$$

$$= \begin{bmatrix} r_{t+1}^{Mom} \\ r_{t+1}^{Val} \\ r_{t+1}^{Mkt} \\ w_t^{Mom} \\ w_t^{Val} \\ \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix}' \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} r_{t+1}^{Mom} \\ r_{t+1}^{Val} \\ r_{t+1}^{Mkt} \\ w_t^{Mom} \\ w_t^{Val} \\ \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} r_{t+1}^{Mom} \\ r_{t+1}^{Val} \\ r_{t+1}^{Mkt} \\ w_t^{Mom} \\ w_t^{Val} \\ \Delta w_t^{Mom} \\ \Delta w_t^{Val} \end{bmatrix}$$

$$= z_t' R z_t - 2 S z_t$$

- Transaction costs are such that a 10% change in the allocation to value costs 20bp of return and a 5% change costs 5bp of return.

An Illustration

- We shall calculate the optimal portfolio allocations for our earlier example. We shall set $\theta=20$.

- We shall compare our optimal dynamic portfolio to one calculated from the solution to the static problem of finding the optimal portfolio w_t given an initial portfolio w_{t-1} that maximises the Legrangian

$$L = \left(w_t' r^e_{t+1|t} - (w_t - w_{t-1})' T (w_t - w_{t-1}) \right) - \frac{\theta}{2} w_t' \hat{\Omega} w_t$$

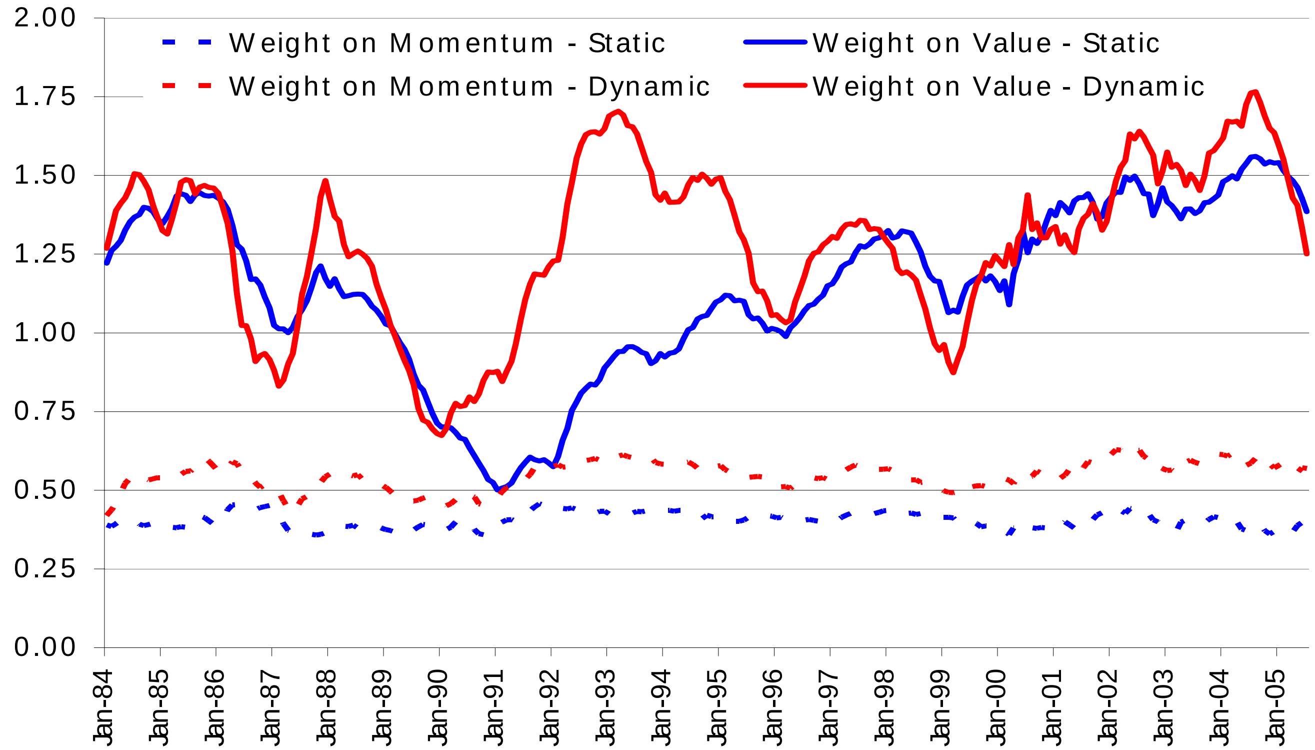
where

$$r^e_{t+1|t} = \begin{bmatrix} r^{Mom}_{t+1|t} \\ r^{Val}_{t+1|t} \\ r^{Mkt}_{t+1|t} \end{bmatrix} \quad \text{and } \hat{\Omega} = Var(e_t) \;\; [= \Omega(5:7, 5:7)]$$

subject to the weight on the market being equal to 1.

A Comparison of the Static and Dynamic Approaches

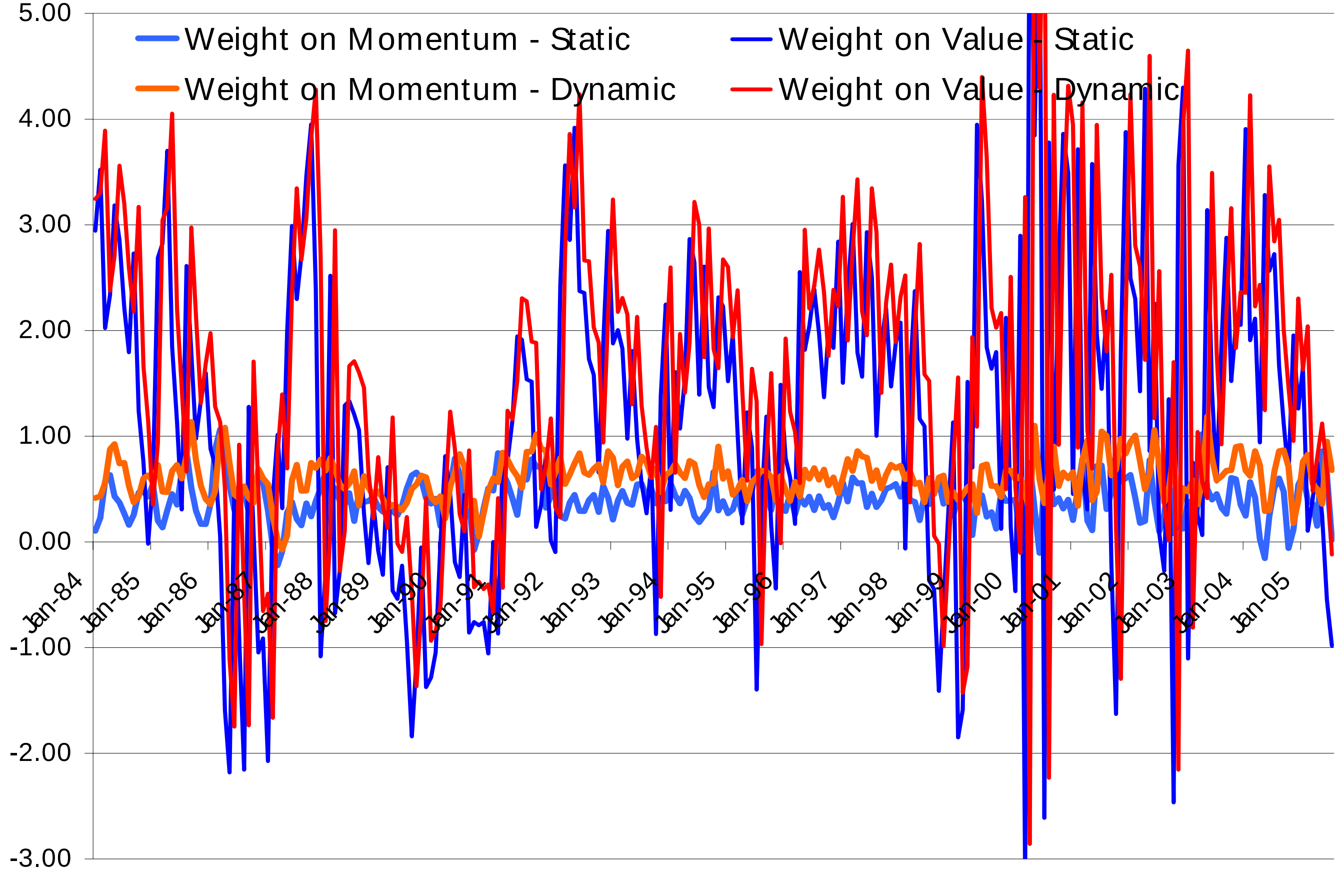
- We compare for the optimal portfolio weights over our data sample



1. The dynamic solution gives a far greater weight to Momentum
2. The dynamic solution plays value far harder during the value market 1992-1996

Deconstructing the solution - Hedging

- Set transactions costs to zero. Differences due to Hedging



1. Dynamic solution increases position on momentum as a hedge against value
2. Dynamic solution increases position on value as a hedge against the market.

Conclusions

- We have phrased the dynamic portfolio construction problem in the risk sensitivity framework of Whittle 1991. This framework is very flexible as allows us to model:
  - Forecast decay rates
  - Forecast uncertainty
  - Correlation in forecasts
  - Transaction costs – both implicit and explicit
  - Portfolio Hedging motives
- We have illustrated the problem on both a long-term strategic asset allocation problem and a shorter term value/momentum trading strategy.