Active Equity Portfolio Allocation Sensitivity and Modern Robust Risk Models

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Plan of the Talk

1. **Provide sharp error analysis (performance bounds) of the fundamental mean-variance efficient portfolio selection problem, exhibit/explain uncertainty amplification**

   (10 slides)

2. **Introduction to a new (non-stochastic) risk model and numerical simulation (out of sample) results comparing many contemporary methods**

   (14 slides -- *the good stuff!*)

Some Background Material

- **Jobson and Korkie (1980) -- Approximate asymptotic series**
- **Michaud (1989) -- Economic/technical treatment**
- **Best and Grauer (1991) -- Sensitivity analysis for quad. MV optimization**
- **Chopra and Ziemba (1993) -- Empirical studies**
- **Britten-Jones (1999) -- Technical analysis**
- **Kritzman (2006) -- Empirical studies**

First Motivating Example

Exhibit 1 a
(Kritzman, 2006, Historical Data)

Asset	Average Historical Return	Historical Standard Deviation	Expected Return	Correct Portfolio Weights	Incorrect Portfolio Weights	Relative Error
Australia	5.97%	26.55%	6.79% (+1%)	2.46%	11.44%	8.98%
Canada	5.94%	21.81%	4.94% (-1%)	0.74%	0.00%	0.74%
France	6.11%	24.54%	7.11% (+1%)	0.45%	17.72%	17.27%
Germany	6.06%	25.43%	5.06% (-1%)	2.42%	0.00%	2.42%
Japan	6.28%	26.26%	7.28% (+1%)	19.42%	25.19%	5.77%
Switzerland	5.61%	20.47%	4.61% (-1%)	13.40%	0.00%	13.40%
UK	5.92%	21.13%	6.92% (+1%)	8.27%	32.56%	24.29%
US	5.70%	17.21%	4.70% (-1%)	52.85%	13.09%	39.76%
Total misallocation						56.32%

Motivation Continued

Exhibit 1 b
(Kritzman, 2006, Asset Correlation Matrix)

Asset	Austria	Canada	France	Germany	Japan	Switzerland	UK	US
Australia	1.00							
Canada	61.01	1.00						
France	39.96	47.93	1.00					
Germany	37.43	44.91	73.93	1.00				
Japan	34.26	34.30	40.63	32.36	1.00			
Switzerland	40.01	46.62	63.92	68.22	40.68	1.00		
UK	55.45	57.83	59.88	54.59	42.04	60.81	1.00	
US	48.79	73.87	54.07	52.01	31.33	53.82	61.20	1.00

Bottom Line: Large asset correlations (across time) are larger, i.e., assets are more linearly dependent, hence the historical covariance matrix will be poorly conditioned.

Second Motivating Example

Exhibit 2a
(Kritzman, 2006)

Asset	Average Historical Return	Historical Standard Deviation	Expected Return	Correct Portfolio Weights	Incorrect Portfolio Weights	Relative Error
Stocks	9.00%	17.21%	8.00% (-1%)	61.42%	60.05%	1.37%
Bonds	4.00%	9.26%	5.00% (+1%)	0.00%	2.35%	2.35%
Cash	2.00%	.063%	3.00% (+1%)	0.00%	0.00%	0.00%
Commodities	8.00%	18.85%	7.00% (-1%)	38.58%	37.59%	0.99%
Total misallocation						2.35%

Exhibit 2b
(Kritzman, 2006, Asset Correlation Matrix)

Asset	Stocks	Bonds	Cash
Stocks	1.00		
Bonds	18.01	1.00	
Cash	8.66	11.77	1.00
Commodities	-7.02	1.02	4.54

Bottom Line: Assets are less (than before) linearly dependent, hence the historical covariance matrix will be better conditioned.

Markowitz Mean-Variance Portfolio Selection

Goal:

$$\min_{w \in \mathbb{R}^N} w' \Sigma w, \quad \Sigma \in \mathbb{R}^{N \times N}, \quad \text{subject to:} \quad w'\mu = \sum_{i=1}^{N} w_i \mu_i, \quad \mu_i = \mathrm{E}[r_{i,t}]$$

What happens when the return data are "inexact?"

$$\tilde{r}_{i,t} = r_{i,t} + \underbrace{\delta r_{i,t}}, \quad i = 1, 2, \cdots, N$$

- **Statistical Outliers** **(return = true_data + "market exaggeration")**
- **Interpolation/Extrapolation Errors**
- **Improper Accounting/Reporting**

Analytical Misallocation Bounds (Controls)

$$\text{LB} \leq \frac{\|\tilde{w} - w\|_{l^2(\mathbb{R}^N)}}{\|w\|_{l^2(\mathbb{R}^N)}} \leq \text{UB}$$

where,

$$\text{LB} = \left\| \frac{(I + \Sigma^{-1}\delta\Sigma)^{-1}\Sigma^{-1}(\mu + \delta\mu)}{e'(I + \Sigma^{-1}\delta\Sigma)^{-1}\Sigma^{-1}(\mu + \delta\mu)} - \frac{\Sigma^{-1}\mu}{e'\Sigma^{-1}\mu} \right\|_{l^2(\mathbb{R}^N)} \frac{|e'\Sigma^{-1}\mu|}{\|\mu\|_{l^2(\mathbb{R}^N)}} \sigma_N$$

(Minimum error that MUST be present in the result)

$$\text{UB} = \left\| \frac{(I + \Sigma^{-1}\delta\Sigma)^{-1}\Sigma^{-1}(\mu + \delta\mu)}{e'(I + \Sigma^{-1}\delta\Sigma)^{-1}\Sigma^{-1}(\mu + \delta\mu)} - \frac{\Sigma^{-1}\mu}{e'\Sigma^{-1}\mu} \right\|_{l^2(\mathbb{R}^N)} \frac{|e'\Sigma^{-1}\mu|}{\|\mu\|_{l^2(\mathbb{R}^N)}} \sigma_1$$

(Maximum error that CANNOT be exceeded in the result)

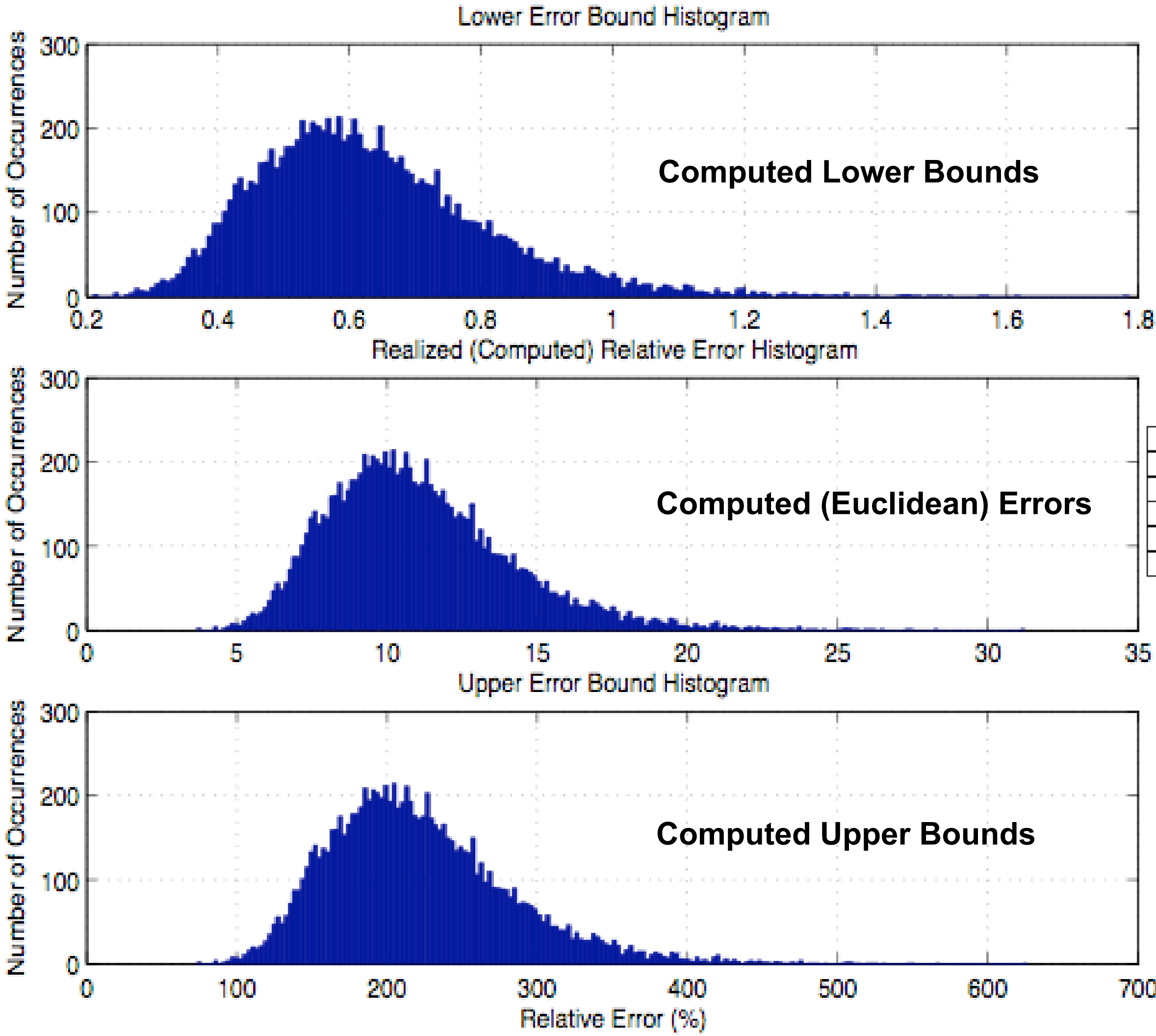
Some Contemporary Mitigating Implementations

• Statistical (covariance) shrinkage, e.g., Ledoit covariance estimation (Ledoit, 1994)

• Bayesian estimation, e.g., boot-strapping (Efron, 1979)

• Robust optimization under uncertainty, e.g., penalized optimization (Many contributors…Dong Shaw 3/26/2008)

Realized and Bounding Errors (5% Additive Noise)*

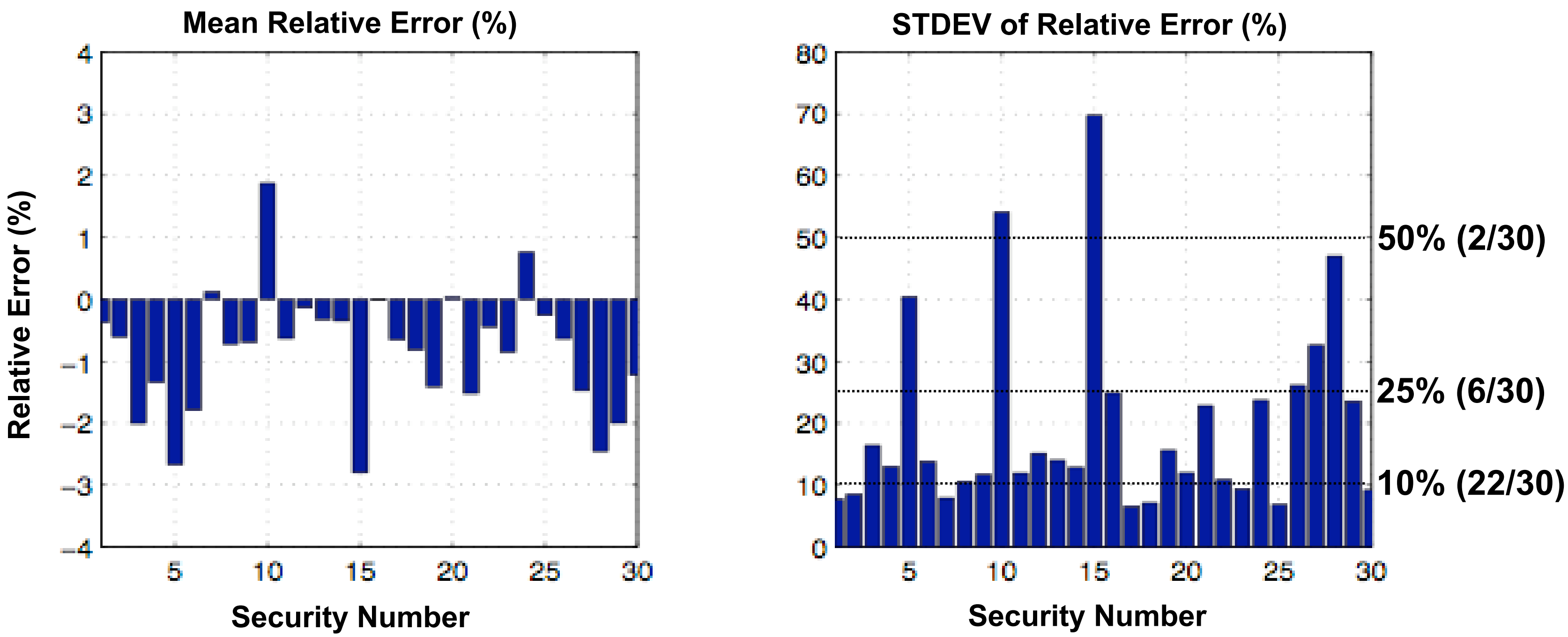


Security Information

Security Tickers 1-5	GE	XOM	MSFT	C	PFE
Security Tickers 6-10	JNJ	WMT	BAC	INTC	AIG
Security Tickers 11-15	PG	MO	JPM	CSCO	IBM
Security Tickers 16-20	CVX	KO	WFC	VZ	DELL
Security Tickers 17-25	PEP	HD	WB	TWX	AMGN
Security Tickers 26-30	COP	ABT	MRK	AXP	LLY

*** Data Source: Dow 30, 8/1996-8/2006 (Sampled Monthly)**

Misallocation Sample Statistics (5% Additive Noise)*



* Data Source: Dow 30, 8/1996-8/2006 (Sampled Monthly)

Generalization: Constrained Misallocation Bounds

1) Solve,

$$\nabla_{(\mathcal{M},\lambda)}\left[w'\Sigma w - \lambda(w'\hat{\mu} - \mu^*)\right] = 0$$

2) Solve,

$$\nabla_{(\mathbb{R}^n,\lambda)}\left[w'\Sigma w - \lambda(w'\hat{\mu} - \mu^*)\right] \cdot h^* \leq 0$$

3) Sensitivity analysis (Best and Grauer),

$$\|w - w^*\| \leq C\sigma_N^{-1}\left(1 + \frac{\sigma_1}{\sigma_N}\right)$$

Alternative Risk Assessment

Variance (Traditional, Industry Standard)

$$\mathrm{Var}(X) = \mathrm{E}\left[(X-\mu)^2\right] \quad \Rightarrow \quad \frac{1}{N-1}\sum_{i=1}^{N} |x_i - \mu|^2$$

Total Variation (Robust Alternative)

$$\mathrm{TV}(f)|_{[a,b]} = \int_a^b |f'(t)|dt \quad \Rightarrow \quad \sum_{i=1}^{N-1} |x_{i+1} - x_i|$$

Subadditivity Property (MV-Keystone)

Variance (Traditional, Industry Standard)

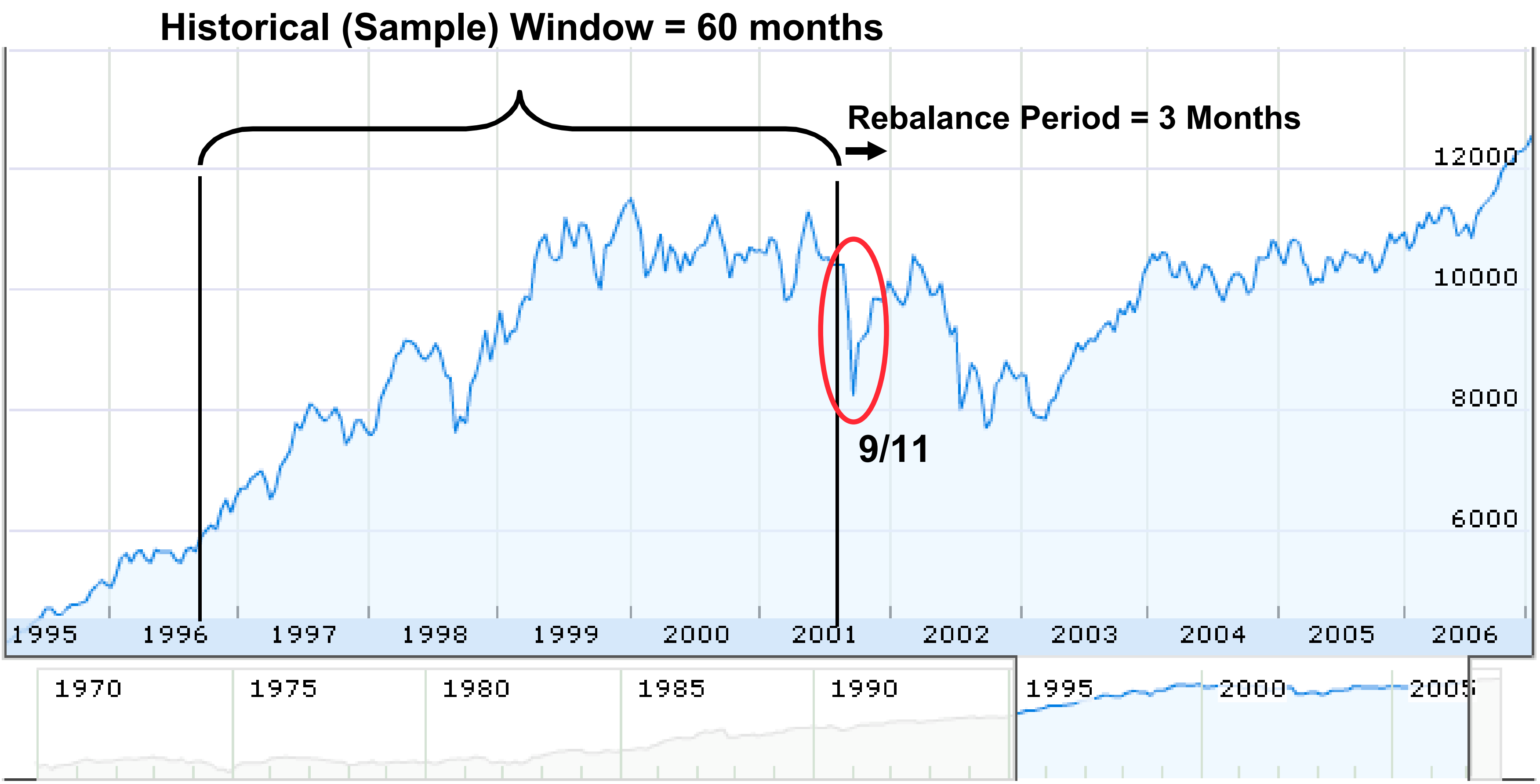
$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\mathrm{Cov}(X,Y) \leq \mathrm{Var}(X) + \mathrm{Var}(Y) \quad \Leftrightarrow \quad \mathrm{Cov}(X,Y) > 0$$

Total Variation (Robust Alternative)

$$\mathrm{TV}(X+Y) \leq \mathrm{TV}(X) + \mathrm{TV}(Y), \quad \text{since} \quad |x+y| \leq |x| + |y|$$

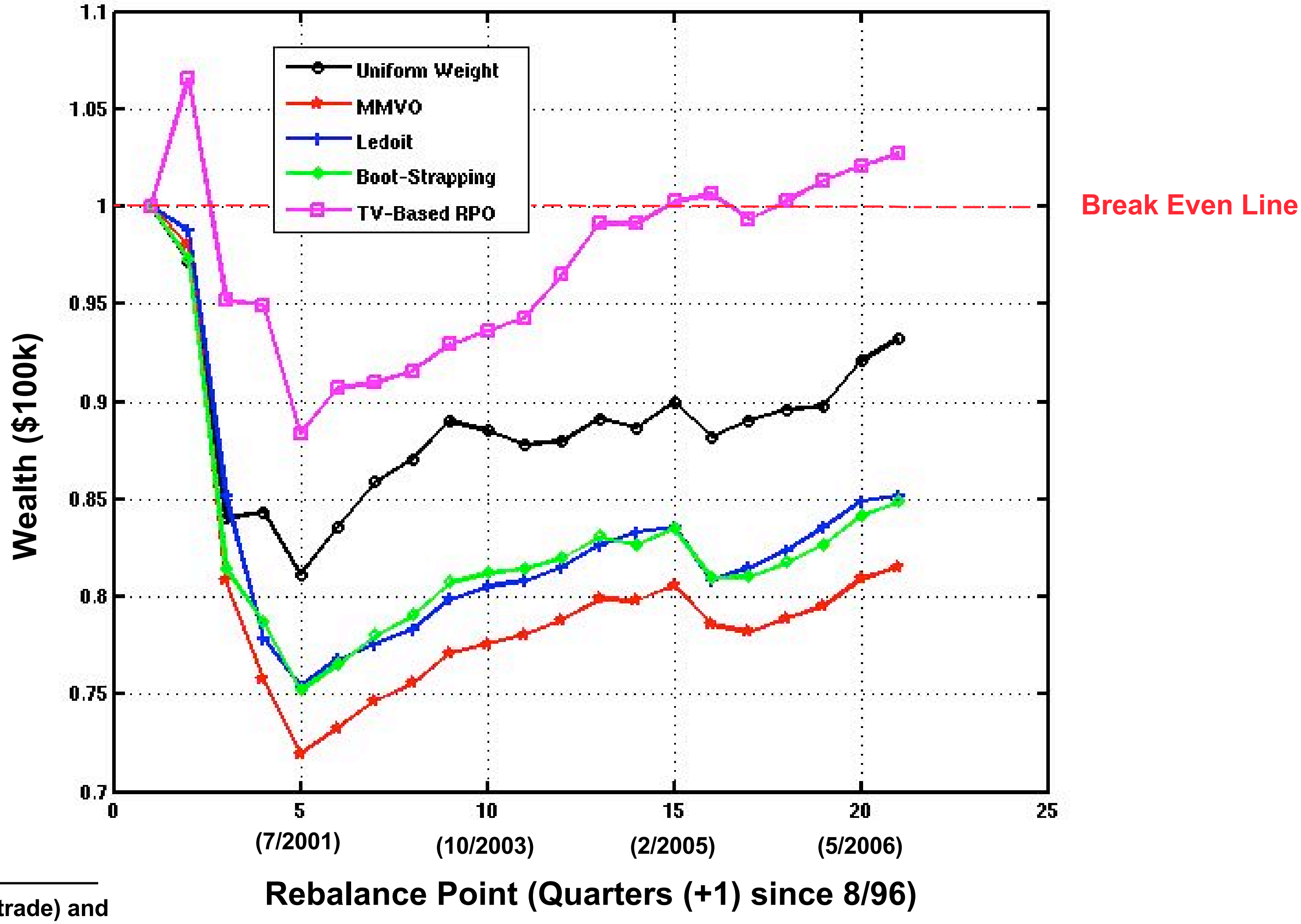
Bottom Line: Total Variation is ALWAYS subadditive, and a more natural (economic) measure -- outlier mitigation!!

Dow 30 Performance Spanning 8/96-8/06*



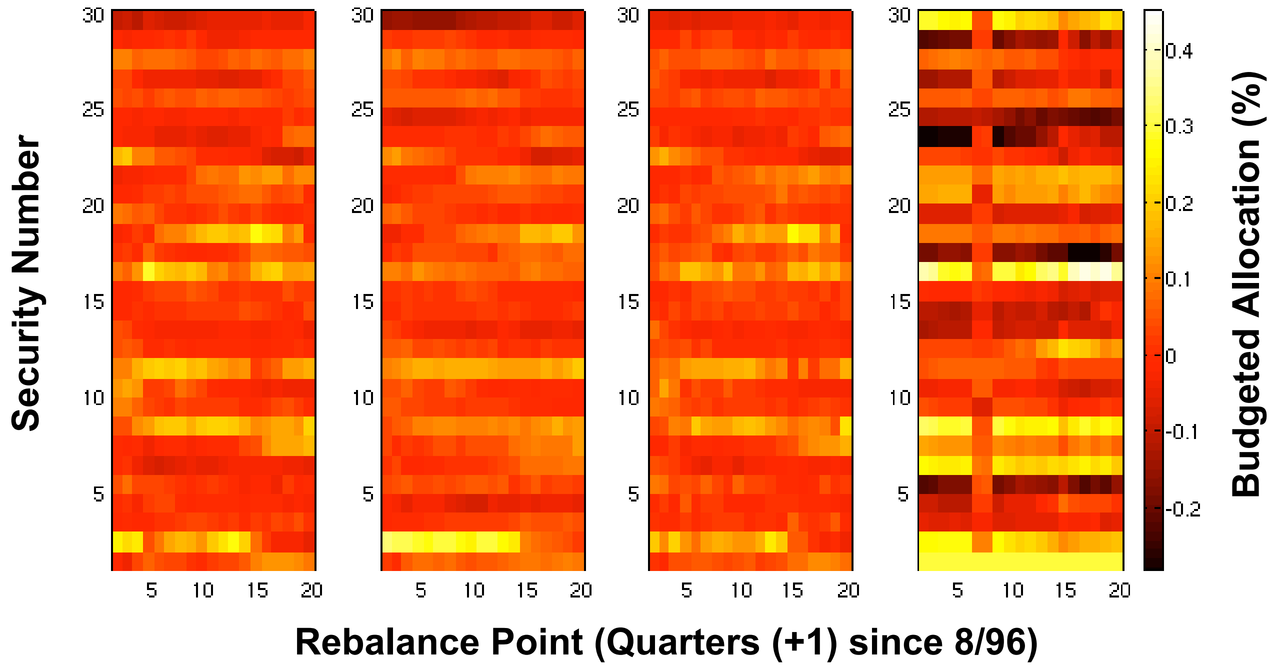
* Data Source: Google Finance

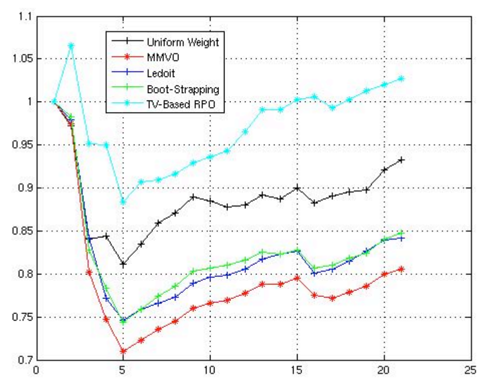
130/30 Wealth Generation* (Downside Risk)



*Transaction fees ($10/trade) and capital gains taxes (flat 30%) included

130/30 Wealth Generation Portfolios

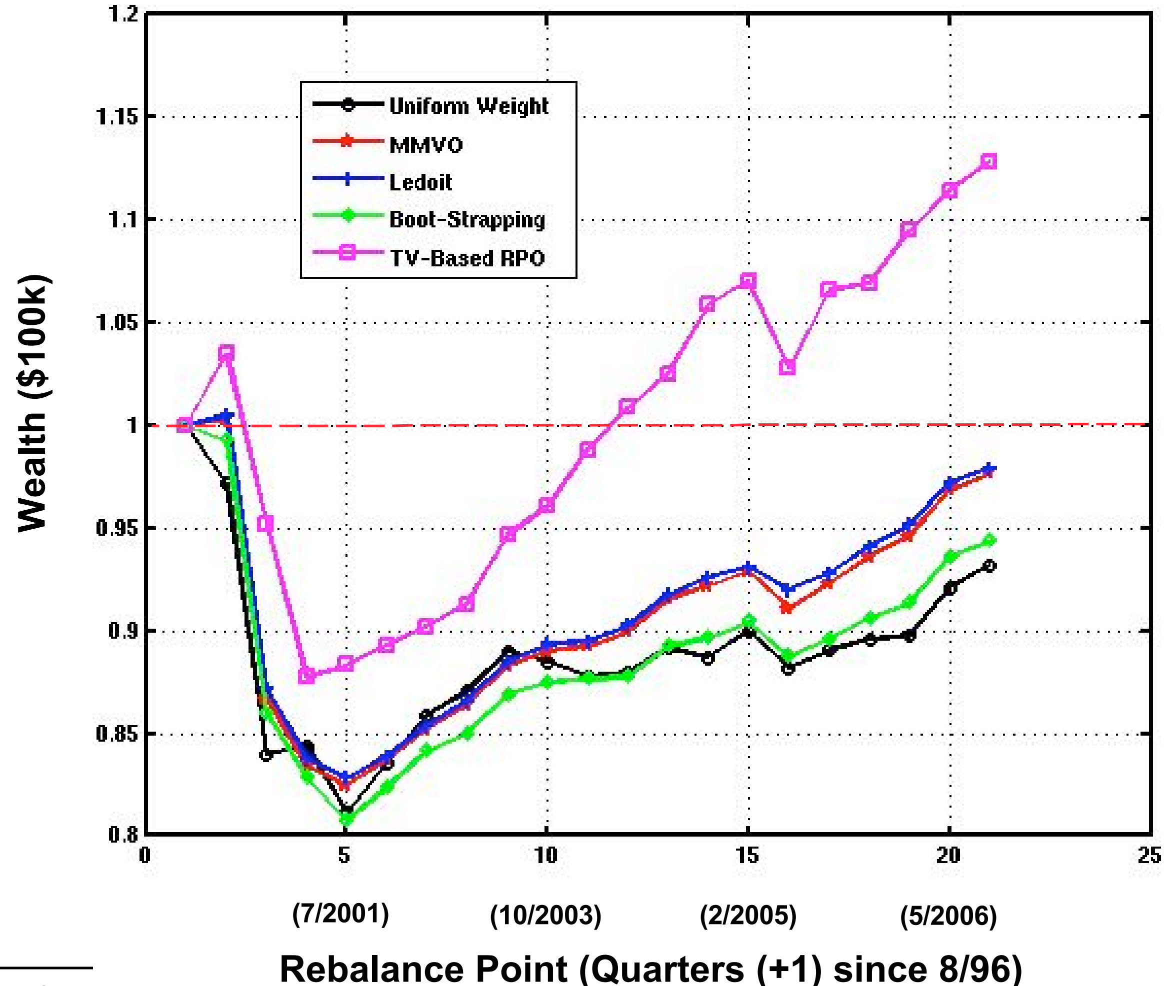




Active 130/30 Allocation Performance Statistics

Allocation Method	100% In the Money Occurrences	90% In the Money Occurrences	Median Wealth ($)	Total 5 Year Gain (Loss) (%)
TV-Based RPO	7/20	19/20	99,134	2.36
Uniform Weighting	0/20	4/20	88,679	(6.77)
Ledoit Shrinkage	0/20	1/20	81,478	(15.07)
Boot-Strapping	0/20	1/20	81,371	(14.97)
MMVO	0/20	1/20	77,564	(19.68)

Long-Only (100/00) Wealth Generation Results*



*Transaction fees ($10/trade) and capital gains taxes (flat 30%) included

100/00 Allocation Performance Statistics

Allocation Method	100% In the Money Occurrences	90% In the Money Occurrences	Median Wealth ($)	Total 5 Year Gain (Loss) (%)
TV-Based RPO	11/20	17/20	100,872	12.83
Uniform Weighting	0/20	2/20	88,679	(6.77)
Ledoit Shrinkage	1/20	13/20	91,768	(2.06)
Boot-Strapping	1/20	12/20	89,882	(5.10)
MMVO	1/20	13/20	91,069	(2.35)

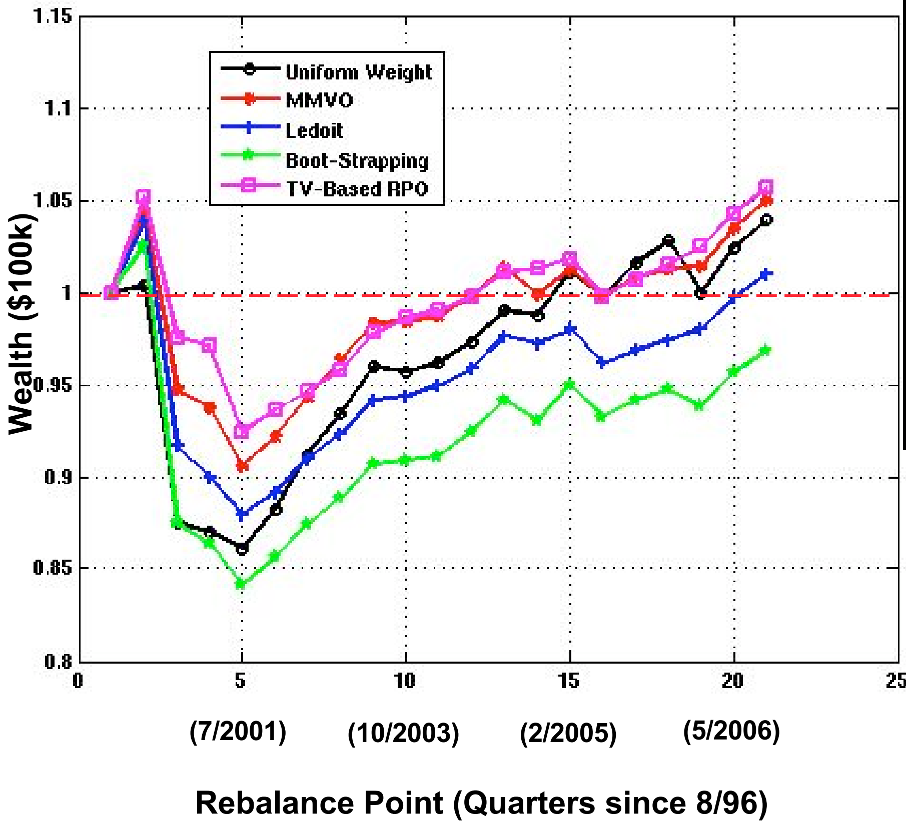
100/00 Allocation Performance Statistics

(300 Random Back-Tests: 30 Equity Assets Drawn From the S&P 500)*

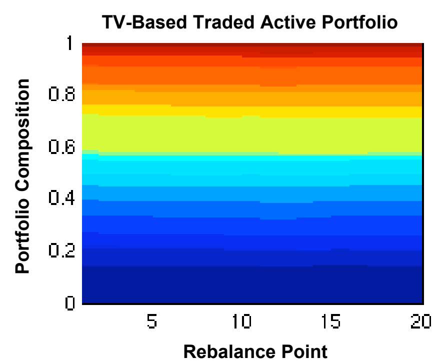
Allocation Method	Mean Sample Wealth ($) Initial Investment $100,000	Relative TV-Based RPO Gain (Basis Points)
TV-Based RPO	101,725 (1,725)	N/A
Uniform Weighting	93,091 (6,909)	927
Ledoit Shrinkage	94,213 (5,787)	797
Boot-Strapping	95,108 (4,892)	695
MMVO	94,734 (5,266)	737

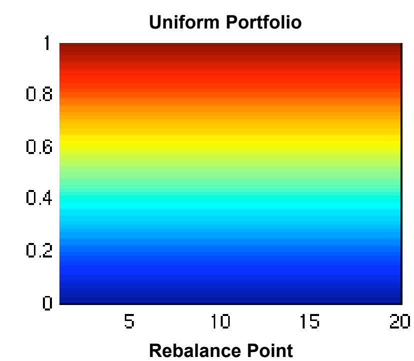
*Transaction fees ($10/trade) and capital gains taxes (flat 30%) included, 60 month window, quarterly rebalancing, dates: 08/96-08/06

Long-Only (100/00) 100 Assets (1st 100 S&P500)



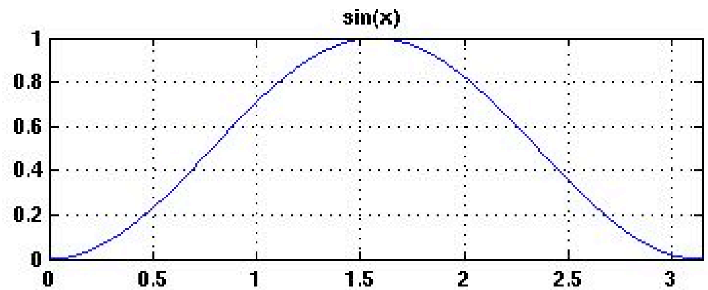
Allocation Method	100% In the Money Occurrences	90% In the Money Occurrences	Median Wealth ($)	Total 5 Year Gain (Loss) (%)
TV-Based RPO	12/20	20/20	99,774	5.64
Uniform Weighting	9/20	17/20	98,764	3.89
Ledoit Shrinkage	3/20	18/20	96,150	0.97
Boot-Strapping	2/20	15/20	93,010	(2.90)
MMVO	12/20	20/20	98,815	4.98



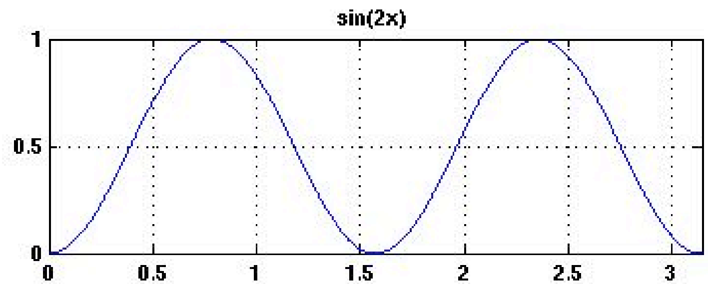


*Transaction fees ($10/trade) and capital gains taxes (flat 30%) included

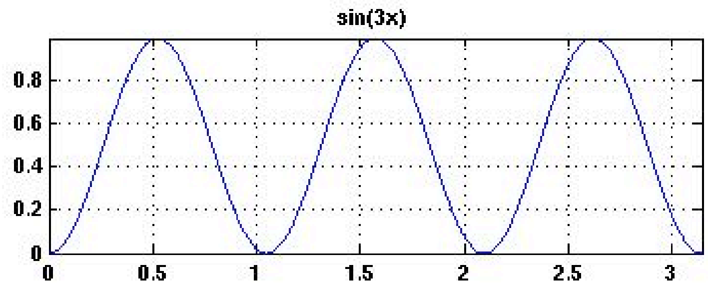
Why Does TV Outperform The Others*?



Mean = 0.0
Var = 1.5708e0
TV = 6.3698e2



Mean = 0.0
Var = 1.5708e0
TV = 1.2740e3



Mean = 0.0
Var = 1.5708e0
TV = 1.9110e3

$$\begin{aligned}\mathrm{Var}\left(\sin(kx)|_{[0,\pi]}\right) &= \int_0^{\pi} \sin^2(kx)dx \\ &= \|\sin(kx)\|^2_{L^2([0,\pi])} \\ &= \frac{\pi}{2}, \quad k = 1, 2, \cdots\end{aligned}$$

$$\begin{aligned}\mathrm{TV}\left(\sin(kx)|_{[0,\pi]}\right) &= k\int_0^{\pi} |\cos(kx)|dx \\ &= \left\| \frac{d}{dx}\sin(kx)\right\|_{L^1([0,\pi])} \to \infty, \quad k \to \infty\end{aligned}$$

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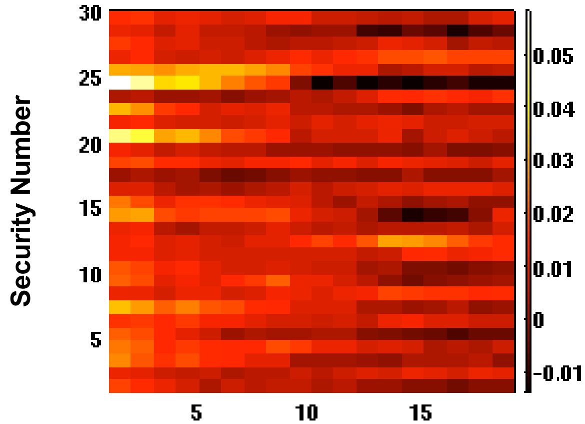
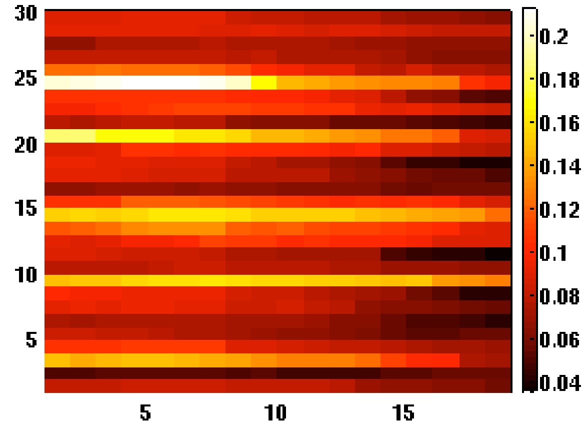
* **Remark: Standard Deviation is NOT a so-called "*Measure of Deviation*", hence it is of little surprise it should lead to suboptimal portfolio results.**

Running Dow 30 Statistics*

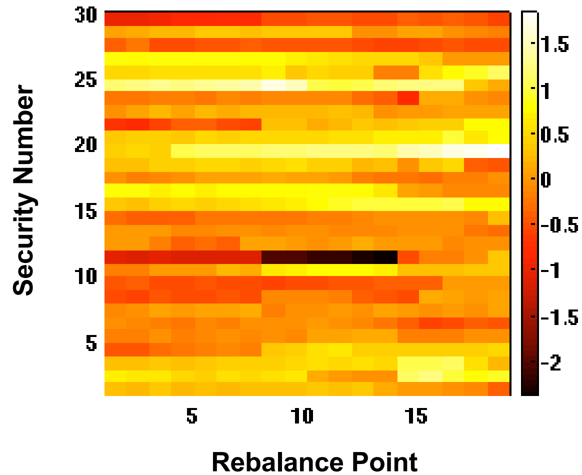
Historical Moving Mean



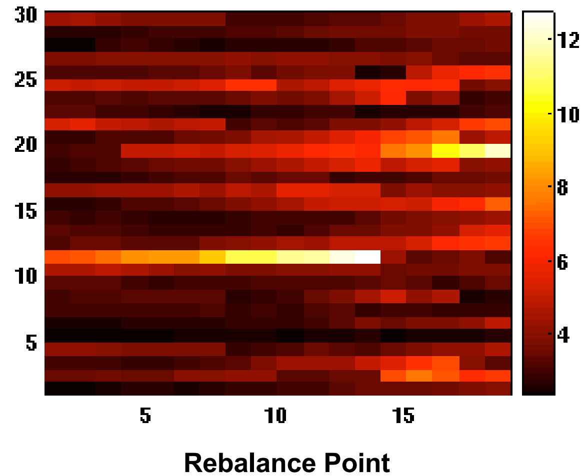
Historical Moving STDEV



Historical Moving Skewness

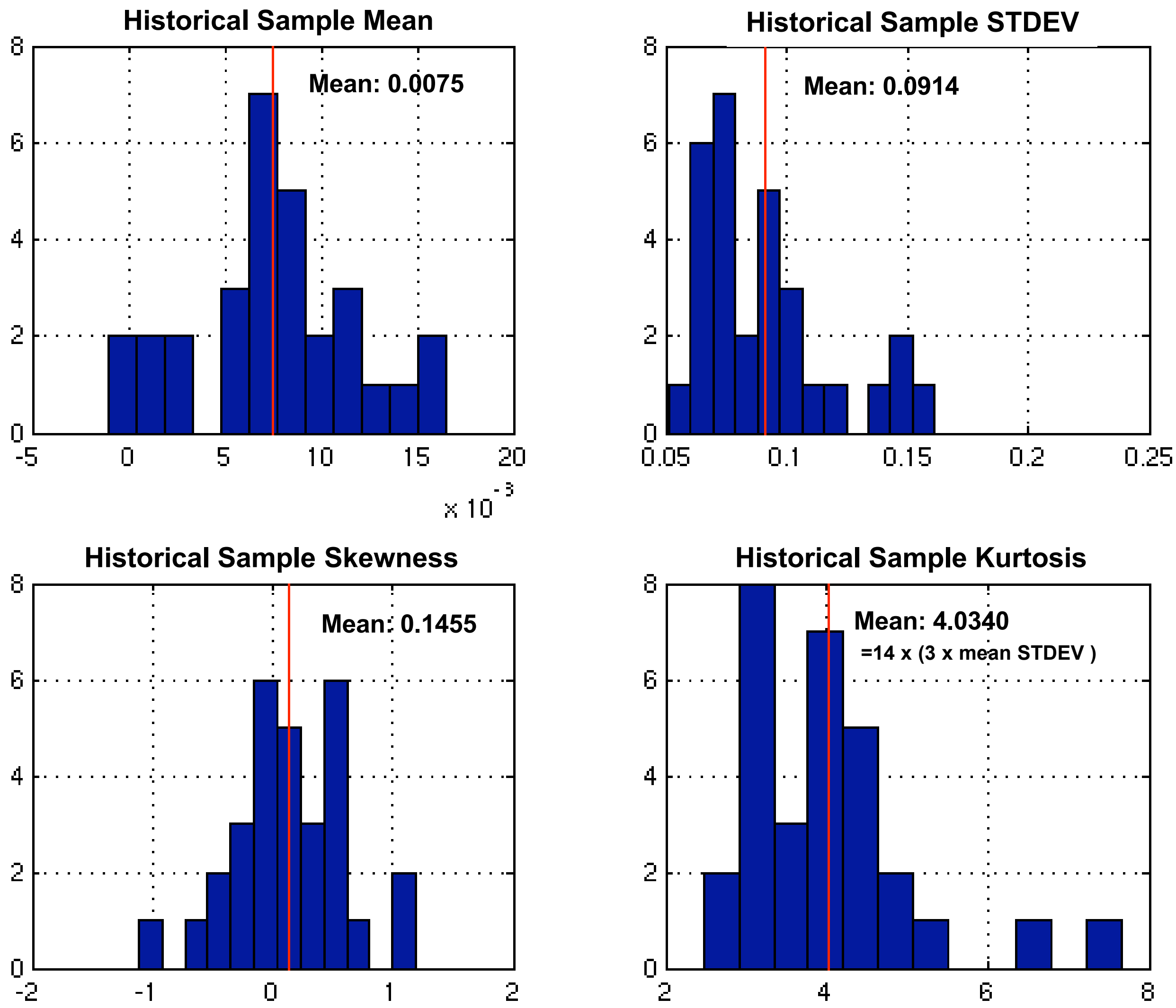


Historical Moving Kurtosis



*Rebalance Point
(Quarters (+1) since 08/1996)

Historical Dow 30 Statistics*



*Rebalance Point
(Quarters (+1) since 08/1996)

Summary

- MMVO performance under uncertainty can be analytically characterized and sharply controlled -- the theory of ill-posed inverse problems brings <u>much</u> to bear on modern quantitative finance

- Many current regularization (stabilization) techniques can improve the (highly variable) performance of MMVO with imperfect data

- MMVO, and stabilized versions of it, can be surpassed in performance in large measure (>600 basis points on 10 year SP500 average) with insightful novel techniques such as those (risk-) based on Total Variation (non-stochastic)

- Equity security data are often <u>not</u> normally distributed, <u>nor</u> stationary in time

References

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[6] Kusiak, S., *Sharp error analysis in unconstrained Markowitz mean-variance efficient portfolio selection*, In review