

The Discretionary Wealth Hypothesis in an Arbitrage-Free Term Structure Approach to Asset-Liability Management

Dan diBartolomeo
Northfield Information Services

Introduction

Traditional approaches to asset-liability management have evolved substantially in recent years. Unfortunately, even the sophisticated, multi-period approaches in common use neglect important features of the underlying economic problem. This paper describes a new approach to asset-liability management that combines four key elements, one of which is quite new to the finance literature.

There are several key benefits to this technique. First, it estimates the present value of future consumption liabilities in a fashion that is consistent with current market conditions, is arbitrage free, and yet does not imply an absolute guarantee of liability fulfillment in cases where no such guarantee is desired. The second key benefit of this technique is that it dynamically reallocates assets over time in a sensible way, based on the “Discretionary Wealth Hypothesis” of Wilcox (2003), and yet is mean-variance optimal as defined in Markowitz (1952) at each moment in time. This approach maximizes the median, rather than the expected value of surplus in a way that more realistically represents investor utility. Finally, we will show that our combined technique is equally suitable as an ALM technique for both institutions and households

This technique also explicitly addresses the existence of transaction costs, and other frictions within the multi-period process, as opposed to the single period assumptions underlying Markowitz.

Key Elements of the Method

Our approach uses a number of building blocks. The first is to use the binomial tree approach from diBartolomeo (1997) to forecast the entire distribution of the surplus between assets and liabilities for all future periods. Each node of the binomial tree represents a potential future state for discount rates and expected returns on assets, and hence the expected value of surplus. In each of these possible future states, we use the “Discretionary Wealth Hypothesis” from Wilcox (2003) to determine the optimal degree of mean-variance risk aversion for the investor. With our determined level of risk aversion, we can apply traditional Markowitz mean-variance optimization to find the

optimal asset allocation for each future state, both across time and across the path dependent scenarios within the binomial tree. To account for the multi-period nature of the process we will introduce a new approximation to modify the “single period” assumption in Markowitz to more precisely incorporate transaction costs in a multi-period Markowitz approach

Traditional actuarial procedures assumed a single rate for discounting future cash outflows to present value. The discount rate is usually set as equal to the expected investment on assets. Conversely, asset cash flows are priced in financial markets by a “yield curve” or term structure of interest rates that reflects investor preferences for maturities and expectations about future changes in interest rates. Due to this conflict of methods, an entity can have riskless incoming cash flows that would *exactly meet all required outflows* but still appear to have a substantial surplus or deficit. Accounting standards for corporations generally embody the concept of the market value of liabilities (i.e. FASB 87), while standards for units of governments often retain the traditional approach, as described by Minahan (2010). In addition, the potential correlation between asset values and the present value of liabilities is often ignored, or addressed in a primitive fashion by assuming some fixed statistical correlation

Our preference is to assume a lognormal interest rate process in discrete time and model the evolution of short-term interest rates a binomial tree. A convenient model of this type is provided in Fabozzi and Dattatreya (1989). Within this model the discount rate for any event subsequent to any point (node) in the tree can be calculated as the average interest rate over all possible future paths. At any point in the tree the forward (in time) present value of any subsequent cash flow can be calculated by discounting the cash flow over all possible future paths. The “tree” nature of the model allows us to separately consider the present values of assets, liabilities and surplus as probability distributions in their own right.

A key step is to calibrate the tree to an arbitrage free condition. This can be done by “bending” the branches of the tree using a backward induction procedure, until all riskless bonds (i.e. Treasuries) have a present value at the root of the tree equal to the market value. Alternatively, at each node we can change the relative probability of an upward or downward movement in the short-term interest rate, as in Black, Derman and Toy (1990).

It should be noted that creating an arbitrage free term structure as a basis for our process, does not imply that the market value of liability cash flows must be discounted at risk-free interest rates. We can simply define our discount rates for a given cash flow to be as determined within the tree structure plus a spread (e.g. +1%, +2%, etc.). Each possible value of the spread corresponds to a likelihood of not fulfilling the future obligation when required. In the context of personal financial planning, we might choose to discount liabilities associated with future retirement income by the risk free rate, while using a higher (risky) discount rate for a desirable but non-essential consumption event such as making a future charitable donation.

For defined benefit pension funds, we can use the tree to solve for the spread that would equate the present value of liabilities and assets, or equate the present value of liabilities with the present value of liabilities as calculated by whatever actuarial standard is currently employed. We can then convert the resulting spreads to estimate the probability that pension obligations will not be met in full without additional financial contributions from the sponsoring entity.

It should be noted that while the ability to add a spread over the risk free rate across the interest rate tree is a convenient way to reconcile our approach with conventional actuarial or personal financial planning practice, there is a practical danger that unrealistically high discount rates will be used to disguise the reality that many retirement plans, whether collective or individual are seriously underfunded.

To complete our basic model, we will represent the prices process for assets as a second binomial tree that is correlated with the interest rate process. Asset returns in each period are the sum of a drift term (risk premium), an effect from correlation with the interest rate process and a noise term. The required estimation is typically done using vector auto-regression techniques as used in Campbell and Viciara (1999). The correlation of the two trees can be represented geometrically as in Margrabe (1978).

Assuming a fixed asset allocation, we can now get an expected value for surplus at any particular node of the tree, and hence the expected value of the surplus distribution at any point in time. What if the asset allocation were not fixed? Since we can project asset returns all along the binomial tree, we can change asset allocation at each node of the tree with no loss of generality. We just have to start projecting asset values at the root of the tree and work outward, just as we work from the branches inward to estimate the present value of liabilities. We propose to change asset allocation dynamically over time, using Markowitz mean-variance optimization, *but allowing state dependent risk aversion that varies with both time and the relationship between assets and liabilities*. One key assumption is that all investment assets may be liquidated to fund consumption, but are subject to non-zero transaction costs. We propose that the general concepts of Markowitz mean-variance efficiency be adhered to, but modified slightly to reflect a multi-period process in discrete time.

The Markowitz MPT says that an investor can form an *efficient frontier* of differently composed portfolios. Each portfolio on the frontier has the maximum return for a given level of risk and also has the minimum risk for a given level of return. The time dimension of the process is defined as a single period of indefinite length. However, the theory itself says nothing about how an investor should go about choosing which portfolio on the efficient frontier to actually hold.

Most finance literature assumes that investor's have some form of quadratic or log-wealth utility function, as described in Levy and Markowitz (1979).

$$U = \alpha - \lambda\sigma^2$$

Where

U = investor utility

α = expected periodic returns

σ = expected standard deviation of periodic returns

λ = investor risk aversion

A financial professional might be well to ponder the problematic nature of asking a passer-by on the street “What is your risk tolerance parameter in mean-variance space?” Investors constantly use terms of art such as “conservative” or “aggressive” to describe their posture without any quantification of what that means. This ambiguity results muddles the reason why investors change their portfolio composition over time. A portfolio might change because expectations have changed about the future returns or risks of available assets, or the portfolio might change because the investor’s risk aversion has changed, probably without being recognized or consciously done.

Discretionary Wealth

One old adage regarding the management of financial risk says simply that investors should not put more money at risk than they can afford to lose. A useful formalization of this idea is provided in the “Discretionary Wealth Hypothesis” from Wilcox (2003). This concept is equally applicable to the asset-liability concerns of both institutions and households.

Financial institutions such as insurers and defined benefit pension schemes have long thought in terms of assets, the present value of liabilities and surplus. For households, it is more novel to think of your life as a *balance sheet*, including implied assets such as the present value of future savings, and the present value of the liability for expected consumption expenditures. It should be noted that for pension schemes, the inclusion of implied assets (discounted at a risky rate) in this framework can help mediate how “going concern” value of the future contributions from the sponsoring entity can be accommodated. See Keating (2010) for further discussion on this issue.

Wilcox then derives that the optimal value for λ , the Levy-Markowitz mean-variance aversion for an investor is equal to one half times the ratio of total assets divided by net worth (surplus). For private wealth, Wilcox recommends use of “investable” assets rather than total assets in the numerator, as a way to convey that some assets such as a personal residence might be considered unavailable for liquidation to fund future consumption.

$$\lambda^* = \text{total assets} / (2 * (\text{total assets} - \text{present value of liabilities}))$$

Where

λ^* = the optimal risk aversion

Given that our tree structure can provide us with expected values for assets and liabilities at each node, we can calculate the expectation of λ^* for each future moment in time. An intuitive way to think of this is that our optimal degree of risk aversion will vary through

time for both predictable reasons (e.g. getting close to retirement age) and unpredictable reasons (e.g. market volatility).

The important outcome of this process is that allowing time variation in λ maximizes the expected median of future wealth, rather than the mean. The effect is a dynamic asset allocation process, somewhat similar to a strategy of portfolio insurance. A good overview of portfolio insurance strategies can be found Bouye (2009). You are increasing aggressiveness when you can afford to do so and taking a more conservative posture when you must as a matter of prudence. Note that these changes only impact your risk aversion level. Actual changes in portfolio composition must also reflect changes in capital market expectations, and expected trading costs.

It is important to highlight that dynamic asset allocation strategies do introduce path dependence into terminal wealth values. Under the approach presented here, if investors experience a period of positive returns followed by a period of negative returns, the resulting wealth will generally be different than if the sequence had been negative returns followed by positive returns. However, the introduction of path dependence is nothing new for individuals. We all live our lives by waiting to cross a street when a car is coming, carrying an umbrella when it's raining, or buying a little extra when a favorite food is unexpectedly on sale at the market.

Also, this form of dynamic allocation may be pro-cyclical in that investors will normally switch from high risk assets (stocks) to low risk assets (bonds, cash) when the stock market declines. Such induced selling of stocks may increase the volatility of equity markets, if carried out by large institutions. It is less likely that even widespread action by retail investors would have a significant effect.

The discretionary wealth approach can also incorporate uncertainty in the balance sheet formation using the Bayesian logic provided in Wilcox and Fabozzi (2009). For example, as individuals we don't know how long we'll live or whether our children will require financial support for education. Institutions might choose to examine the impact of inflation on future outgoing cash flows in this way, or incorporate such concerns directly in our tree structure.

A Walk Through the Algorithm

Assuming we have our two calibrated trees and our asset return equation, we can go forward with the process. For discussion purposes, we'll use one year time blocks.

1. Start at the root of the interest rate tree (today). Move forward randomly through the tree until you reach the other end at time equaling thirty years from now. This will give us a series of thirty one-year interest rate scenarios which we might experience.
2. We now discount the newly projected series cash flow liabilities using the series of thirty interest rates along our time path. Note that the projected "then" present value of subsequent liabilities is forecast for each year in the thirty year series. Save this series of thirty projected present values of subsequent liabilities.

3. Now follow the same path along the asset pricing tree, using our return equation to project returns for each year period for each asset class.
4. We now go back to the root of the tree and calculate our initial surplus as the difference between current value of assets and the present value of liabilities. Following the discretionary wealth approach, we calculate the value of λ^* . Using this value and our return expectations for each asset class, we can calculate the initial Markowitz optimal portfolio.
5. We now move to the second point along our interest rate tree, and estimate the value of assets at this node of the tree. This value would be previous asset value plus the change in value due to the first period return, inclusive of both the expected return and a random noise component. We can also subtract any cash outflows from asset values to represent assets that would have to be liquidated to fund consumption.
6. Once we have asset value at the second node on the path, we can calculate a new value for λ^* . With this input and our return expectations at the second node, we can rebalance the asset weights in the portfolio, subject to transaction costs. We repeat this process for nodes three through thirty of the path.
7. Go back and start again at step #1. Recall that since many of the steps in the analysis assume random movements and returns, each trip through the tree represents a different sequence of events. Repeat the entire procedure a large number of times (as many as necessary) to reach the desired confidence interval around the estimates of the parameters of the resultant probability distributions.

As in most matters of random sampling, the standard errors on the parameter estimates are inversely proportional to the square root of the number of trials. Depending on how far into the future we are choosing to project and how precise a result we wish to obtain, a larger number of trials may be required. While this is not usually a problem with today's fast computers, methodologies for reducing the required number of sample paths are presented in Ho (1992) and Beaglehole (1997).

Assuming that we repeated this procedure a thousand times, we would have one thousand series of projected funding surpluses (deficits), each series containing thirty separate projections for each of the next thirty years. If we look at each year, we can assess the **probability distribution** of funding surpluses (deficits) at that future moment in time.

Moving Markowitz Optimization to a Multi-period Process

Our multi-period approach requires a change to the traditional Markowitz assumption of a time being a single long period. If we know that our portfolio will be changing over time, and those changes require transaction costs, we need to weigh the benefits of improvements in expected utility against the trading costs in the right way. Traditional optimization procedures that assume a single period can be substantially improved by incorporating a simple approximation.

This approximation herein is a cruder form of the multi-period optimization heuristic (“MvD”) presented in Markowitz and Van Dijk (2003), but is computationally much more tractable. In Kritzman, Mygren and Paige (2007) the authors are able to test the efficiency of MvD against full dynamic programming for cases up to only five assets, as dynamic programming becomes computationally too burdensome. The authors are able to extend the MvD method to one hundred assets. The technique presented is routinely used for portfolios of many thousands of assets.

Imagine we have a portfolio, P_1 with return α (net of fees and expenses) and standard deviation σ . Our usual utility function would say:

$$U_1 = \alpha - \lambda\sigma^2$$

Now let’s imagine that another portfolio, P_2 has a higher utility, because either the return is higher or the standard deviation is lower. This portfolio has completely different positions than the initial portfolio. Let’s assume that this portfolio has a higher return by positive increment Δ , so:

$$U_2 = (\alpha + \Delta) - \lambda\sigma^2$$

Since U_2 is greater than U_1 , we should be willing to pay some transaction costs to switch from P_1 to P_2 . Now let’s consider a different way to improve our returns. We go back to the manager of Portfolio 1 and ask them to reduce their fees by Δ , so now our revised utility on P_1 is U_{1L} for “lowered fees”.

Notice that U_{1L} and U_2 are equal. So if we invest our money in either P_2 or P_{1L} (after lowering the fees), the expected value of wealth at the end of time is the same. This suggests that we should be willing to pay the manager an upfront fee to lower ongoing management costs equal to the trading costs we would be willing to pay to switch from the initial portfolio P_1 to P_2 . *As long as conditions never change, this is valid.*

However, since P_2 and P_{1L} have different securities, the performance will be different from month to month, even if the long term average return and volatility are identical. So over any finite time horizon, we cannot be sure which of the portfolios will perform better. On the other hand, P_{1L} will always perform better than P_1 over all time horizons, because it is just the same portfolio with lower fees. For P_{1L} the probability of outperforming P_1 is always one. P_2 is guaranteed to be better than P_1 in the long run if conditions don’t change, but the probability that P_2 will actually outperform P_1 over any finite horizon is between one half and one. We call this probability value the “probability of realization” of the utility increase.

Implementation of Multi-Period Optimization

A typical way of accounting for transaction costs in an optimization is to deduct these costs from the objective function, amortizing the expected costs over the expected holding period of the investment.

$$U_1 = \alpha - \lambda\sigma^2 - (\sum_{j=1}^N (\text{abs}(W_{ij} - W_{fj}) * K_j) * M$$

Where

N = the number of assets in the portfolio

W_{ij} = the initial weight of asset J

W_{fj} = the optimal weight of asset J

K_j = the unit cost of trading asset J

M = the amortization constant for portfolio turnover

Amortizing transaction costs over a single period is equivalent to assuming that the probability of realization is always one. Let's assume a single period optimization for a strategy with expected turnover of M% per annum. Put differently, the expected holding period is just 100% / M%. In our multi-period world, we want to amortize by M divided by the probability that the revised portfolio will actually realize a better risk adjusted return over the finite holding period. That is:

$$M^* = M / Z$$

Where

M* is the adjusted amortization rate

Z is the probability of realization

To calculate Z we can use the tracking error (expected volatility of the return differences) between any two portfolios as a standard error on the expected differences in utility. We obtain the tracking error value from whatever model we are using to estimate σ in our utility function. We can then calculate a T-statistic on the expected difference in utility and calculate Z under a one-tailed test and our choice of probability distribution.

In our example above, the value of Z is one for the incremental utility of P_{1L} relative to P₁. This is because the tracking error between P₁ and P_{1L} is zero, as they are the same portfolio with different fee structures.

Conclusions

Traditional asset-liability management for both institutions and households has substantial limitations arising from the distortions caused by overly simplistic methods for discounting liabilities. Our approach resolves several of these issues. By combining the arbitrage free discounting process with the Discretionary Wealth Hypothesis, we allow investors to sensibly employ Markowitz mean-variance analysis over time in way that both anticipates and responds optimally to changes in surplus. We also provide a required adjustment to the usual mean-variance treatment for trading costs to reflect the multi-period nature of the problem.

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