The Sensitivity of Beta to the Time Horizon when Log Prices follow an Ornstein-Uhlenbeck Process

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Presentation Outline

I. Introduction

II. Framework

III. CAPM beta

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Introduction

- One flaw of some current risk models is a wonderful vagueness about the forecasting horizon

- If data are IID, and model is monthly and returns are additive, say, then annual return / forecast can be consistently estimated by multiplying returns by 12, and standard deviation by square root of 12.

- However, even this assumes that the true value is based on parameters defined over some particular frequency. In statistical terms, we have a true model defined over period T, and we have time aggregation and disaggregation.

Introduction

- The way vendors deal with this is vagueness rather akin to that of the analyst.
- So, if you forecast 4% risk, and risk is 4% in a year's time, you are right
- But if risk is 4% at any time, before or after 1 year, you are also right

Introduction

- Wonderfully reminiscent of an analyst's target price

- We shall look at this problem in a bivariate context (CAPM), but we let errors be correlated corresponding to other factors, hence quite general. In fact, this is relevant to any linear factor model based on estimating exposures one factor at a time so covers many cases of screening on regression coefficients

- In risk models, the estimates of risk exposure (betas) are always calculated conditionally and in what follows, we shall consider the evaluation of betas conditional on current time

Introduction

Short Measurement Period of CAPM Beta

- Returns measured over shorter periods should have more information on asset’s risk profile than returns measured over longer periods

- Returns measured over shorter periods suffer from friction in the trading process often appears as autocorrelation in the returns .

- Manifestation of illiquidity, see Dimson (JFE 1979) etc.

Introduction

Intervalling Effect

- Empirically, estimated beta values are systematically changed as the return measurement interval is varied if iid additive assumption violated

- It causes a systematic bias in the performance measures of each security hence the deviation between the theoretical model and the empirical evidence

Introduction

Levi and Levhari (1977)

- They assume iid multiplicative returns

- Illustrates the assumed horizon plays a crucial role in empirical testing in that any deviation from the "true" horizon causes a systematic bias in the beta, a measure of systematic risk

Introduction

Levi and Levhari (1977)

- Levhari and Levy (1977) proved the result that the expected value of the estimated beta of aggressive stocks (Beta greater than one) would increase as the interval increased and hence be over-estimated with the opposite happening for defensive stocks. A positive monotonicity result in time horizon, h

- This has come to be known as the Levi-Levhari (LL) hypothesis

- This is a fantastic result

Introduction

Other Literatures

- Hawawini (JFQA 1980) provided some theoretical results based on weakening the independence assumption, and, like us, considers additive returns

- Kim (1999) also looks at additive returns

- The above results get formulae but not patterns of biases

- Large number of papers look at empirical implications, more recent contributions include Josev, Brook, and Faff (2001) and Diacogiannis and Makri (2008)

Introduction

Other Literature.

- Perron and Vodounou (1997): This is similar to our paper, more about temporary and permanent components of returns. Their notion of investment horizon is designed to correspond to sampling return of different horizons based on non overlapping data, which we wish to separate

- Most importantly, their calculations are unconditional and seem less well suited for understanding the conditional time horizon of risk

Introduction

Contributions of the paper

- To establish a theoretical groundwork that allows for both additive and multiplicative returns, both autocorrelation and trending in the asset and the market
- To allow the length of the measurement interval to be positive and not necessarily an integer valued parameter

II. Framework

Assumption

- We assume log prices follow a bivariate log OU process, which has been used for modeling mean reverting and trending processes

- The bivariate log OU process imposes stronger assumption than Sharpe’s since bivariate log OU process assumes the joint distribution of market and asset returns is normally distributed, whilst Sharpe’s model requires only that the asset returns are conditionally normal, PV (1997) does not assume Sharpe’s structure

II. Framework

Log OU Process

- The equations can be explicitly solved and there are exact solutions for discretized versions of this model

- It allows for stationary (mean-reverting) behavior, random walks and explosive (trending) behavior in either or both variables

- It also produces ARMA models for returns consistent with the structures proposed by Poterba and Summers (1988) and Fama and French (1988)

II. Framework

Return Process

- Assume that the logarithm of the asset prices $\log P_A(t)$ and $\log P_M(t)$ have linear trends $\mu_A(t)$ and $\mu_M(t)$ respectively. We consider the process

$$q_M(t) := \log P_M(t) - \mu_M t \quad (1)$$

$$q_A(t) := \log P_A(t) - \mu_A t \quad (2)$$

II. Framework

Log Price Process

- We call the following the "detrended log price process"

$$dq_A(t) = \left(-\theta_{A1} q_A(t) + \theta_{A2} q_M(t)\right)dt + \sigma_A dW_A(t) \quad (3)$$

$$dq_M(t) = -\theta_M q_M(t)dt + \sigma_M dW_M(t) \quad (4)$$

II. Framework

Log Price Process

- $W_M(t)$ and $W_A(t)$ are correlated Wiener process with correlation coefficient κ $E[dW_M(t)dW_A(t)] = \kappa dt$
- The correlation captures all the missing factors
- There is no reason to believe that either the asset or market return converges faster than the other, we assume $\theta_{A1} \neq \theta_M$
- Treat $\theta_{A1} = -\theta_{A2}$ as a special case of the above

II. Basic Framework

Proposition 1

- The return vector, $R(t+h,t)=\begin{pmatrix} r_A(t+h,t) \\ r_M(t+h,t) \end{pmatrix}=Q(t+h)-Q(t)+\mu$

can be written as:

$$R(t+h,t)=\begin{pmatrix} \mu_A \\ \mu_M \end{pmatrix}h+\begin{pmatrix} e^{-\theta_{A1}t}(e^{-\theta_M h}-1) & \dfrac{\theta_{A2}}{\theta_{A1}-\theta_M}\left(e^{-\theta_M t}(e^{-\theta_M h}-1)-e^{-\theta_{A1}t}(e^{-\theta_{A1}h}-1)\right) \\ 0 & e^{-\theta_M t}(e^{-\theta_M h}-1) \end{pmatrix}\cdot Q(t)$$
$$+\int_t^{t+h}\begin{pmatrix} e^{-\theta_{A1}(t-u)} & \dfrac{\theta_{A2}}{\theta_{A1}-\theta_M}\left(e^{-\theta_M(t-u)}-e^{-\theta_{A1}(t-u)}\right) \\ 0 & e^{-\theta_M(t-u)} \end{pmatrix}\cdot HdW(u) \qquad (5)$$

III. CAPM Beta

- **Asset Return Covariance**

$$Cov\left[r_A(t), r_M(t)\right] = \sigma_{rArM} = \left(\frac{e^{2\theta_M h} - 1}{2\theta_M} - \frac{e^{(\theta_M + \theta_{A1})h} - 1}{\theta_{A1} + \theta_M} \right) \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \sigma_M^2 + \frac{e^{(\theta_M + \theta_{A1})h} - 1}{\theta_{A1} + \theta_M} \sigma_A \sigma_M \kappa$$

➡ conditional on time t, the covariance is linear in the instantaneous covariance. Here $r_A(t)$ refers to additive returns from time t to time t + h

- **Market Return Autocovariance**

$$Cov\left[r_M(t), r_M(t+\tau)\right] = Cov_{rMM\tau} = \left(\frac{e^{2\theta_M h} - 1}{2\theta_M} \right) e^{\theta_M \tau} \sigma_M^2$$

III. CAPM Beta

Proposition 2

- The one period population beta with length of h (h-beta) under non-zero autocorrelation in return series can be expressed as

$$\beta_{A,M}(h) = \left(1 - \frac{e^{(\theta_{A1}+\theta_M)h}-1}{e^{2\theta_M h}-1} \cdot \frac{2\theta_M}{\theta_{A1}+\theta_M}\right)\frac{\theta_{A2}}{\theta_{A1}-\theta_M} + \frac{e^{(\theta_{A1}+\theta_M)h}-1}{e^{2\theta_M h}-1} \cdot \frac{2\theta_M}{\theta_{A1}+\theta_M}\beta_I \quad (10)$$

$$\beta_I = \frac{\sigma_A}{\sigma_M}\kappa$$

III. CAPM Beta

CAPM h Beta

- h-beta is a positive transformation of instantaneous beta β_I

- Instantaneous beta is considered to be underlying or true beta, but mainly because it's a parameter of the true model

- The strength of our approach is that we can assume the true beta to be the beta at any frequency and still carry out an analysis

III. CAPM Beta

CAPM h Beta

$$\beta_{A,M}(h) - \beta_I = \left(1 - \frac{(e^{(\theta_{A1}+\theta_M)h} - 1)}{(e^{2\theta_M h} - 1)} \cdot \frac{2\theta_M}{\theta_{A1} + \theta_M}\right)\left(\frac{\theta_{A2}}{\theta_{A1} - \theta_M} - \beta_I\right) \quad (11)$$

- This gives an exact relationship between the h beta and the instantaneous beta in terms of over and underestimation for different h

- All the parameters can be estimated, we shall discuss the results later

III. CAPM Beta

Monotonicity of Beta with respect to h

- LL proves positive monotonicity under i.i.d multiplicative return assumptions, therefore, any test of the LL model is an implicit test of monotonicity
- With our bivariate log OU model, we can perform an analysis of monotonicity on the measure of systematic risk, the beta

III. CAPM Beta

Proposition 3

- For the stationary case, with $\theta_M > 0$ and $\theta_{A1} > 0$ the derivative

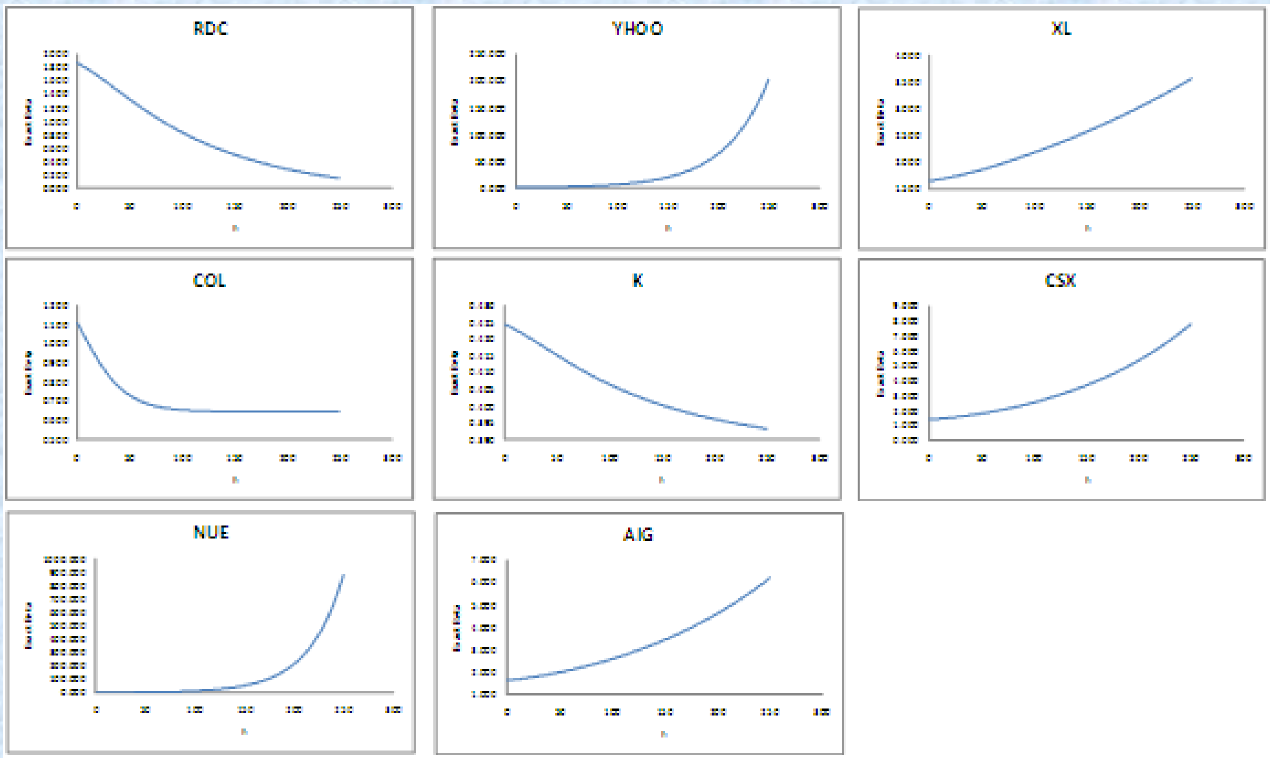
$$\frac{\partial\left(\beta_{A,M}(h) - \beta_I\right)}{\partial h}$$

is monotonic with h

Comment: The monotonicity can be either increasing or decreasing, we give the relevant conditions

III. CAPM Beta

Figure I: Exact Beta vs. h



III. CAPM Beta

Monotonicity of Beta with respect to h

- Whether the exact beta is monotonically increasing or decreasing is governed by the signs of the first and the second term (T1 and T2 hereafter) of equation (11)
- Equation (11) Revisited

$$\beta_{A,M}(h)-\beta_I=\underbrace{\left(1-\frac{(e^{(\theta_{A1}+\theta_M)h}-1)}{(e^{2\theta_M h}-1)}\cdot\frac{2\theta_M}{\theta_{A1}+\theta_M}\right)}_{(T1)}\underbrace{\left(\frac{\theta_{A2}}{\theta_{A1}-\theta_M}-\beta_I\right)}_{(T2)} \quad (11)$$

III. CAPM Beta

Table 1: Monotonicity of Asset Exact Beta

T2 \ T1	$\theta_M > \theta_{A1}$	$\theta_M < \theta_{A1}$
$\beta_I < CV$	T1 > 0 T2 > 0 $\beta_{A,M}(h)$ is monotonically increasing with h	T1 < 0 T2 > 0 $\beta_{A,M}(h)$ is monotonically decreasing with h
$\beta_I > CV$	T1 > 0 T2 < 0 $\beta_{A,M}(h)$ is monotonically decreasing with h	T1 < 0 T2 < 0 $\beta_{A,M}(h)$ is monotonically increasing with h

III. CAPM Beta

Remark 1

- If we let $h \to 0$, it follows that

$$\lim_{h \to 0} \beta_{A,M}(h) = \beta_I$$

- This shows that the instantaneous beta is the limiting value of beta. As the return interval decreases to zero, the value of beta converges to the instantaneous beta

III. CAPM Beta

Proposition 4

- The asset return beta can be approximated with second order Taylor series

$$\beta_{A,M}(h) \approx -\frac{1}{2}\theta_{A2}h + \left(1 + \frac{1}{2}(\theta_{A1} - \theta_M)h\right)\beta_I + O(h^2) \quad (13)$$

We call the value of β_I that makes β_I equal to the order h approximation of $\beta_{A,M}(h)$, the critical value (CV), which is equal to $\theta_{A2}/(\theta_{A1} - \theta_M)$. This is the same as in the exact relationship

III. CAPM Beta

Proposition 4

- It is an immediate consequence of Proposition 4 that the approximate sensitivity of the asset beta with respect to the return measurement interval is

$$\frac{\partial \beta_{A,M}(h)}{\partial h} = -\frac{1}{2}\theta_{A2} + \frac{1}{2}(\theta_{A1} - \theta_M)\beta_I \quad (14)$$

III. CAPM Beta

Proposition 5: The Levi-Levhari Hypothesis applied to the bivariate log OU Model

- Given the assumptions in proposition 4, and when $\theta_{A1} = \theta_{A2}$ so that the critical point is $\theta_{A1} / \left(\theta_{A1} - \theta_{M}\right)$

If , $\theta_M > 0$ CV is > 1

If , $\theta_M < 0$ CV < 1

If , $\theta_M = 0$ CV = 1 (The market efficient case)

III. CAPM Beta

Remark 2

- The O(h) bias on $\beta_{A,M}(h)$ caused by the intervalling effect and the approximate sensitivity of the beta with respect to h can be summarized as

If $\beta_I > \frac{\theta_{A2}}{\theta_{A1}-\theta_M}$, $\beta_{A,M}(h) > \beta_I$: $\beta_{A,M}(h)$ is upward biased and $\frac{\partial \beta_{A,M}(h)}{\partial h} > 0$

If $\beta_I < \frac{\theta_{A2}}{\theta_{A1}-\theta_M}$, $\beta_{A,M}(h) < \beta_I$: $\beta_{A,M}(h)$ is downward biased and $\frac{\partial \beta_{A,M}(h)}{\partial h} < 0$

If $\beta_I = \frac{\theta_{A2}}{\theta_{A1}-\theta_M}$, $\beta_{A,M}(h) = \beta_I$: $\beta_{A,M}(h)$ bias disappears and $\frac{\partial \beta_{A,M}(h)}{\partial h} = 0$

III. CAPM Beta

Comments

- Note that $\theta_{A1} \neq \theta_M$ was assumed

- The intervalling effect on the approximate beta and the speed of the beta convergence to the instantaneous beta depends on the magnitude of θ_{A1}, θ_{A2} and θ_M

III. CAPM Beta

- The result shows that the intervalling effect can cause the beta to be over or underestimated depending on the magnitude of the true beta and the degree of auto-correlation

- It is not in general true that under/over estimation is determined by whether the stock is aggressive or defensive but it is determined by autocorrelation in the stock and the market returns

- Our results show that both positive and negative monotonicity can occur, this differs from LL

IV. Multiplicative Returns

Proposition 6: The LL Theorem for Multiplicative Returns for iid processes

- Given the assumptions of Multiplicative Returns and when θ_{A1} , θ_{A2} and θ_M and the riskless rate are zero,

Case (i): the CAPM holds instantaneously, then

$$Beta(A,M,h) = \frac{\exp((\mu_M)(\beta_I - 1)h + .5((\sigma_A^2(1-\kappa^2)^2)h)(\exp(\beta_I \sigma^2{}_M h) - 1)}{(\exp(\sigma_M^2 h) - 1)} \qquad (18)$$

- If $\mu_M > 0$ and $\beta_I > 1$, we have $Beta(A,M,h) > 1$
  If $\beta_I = 1$, $Beta(A,M,h) = \exp((.5((\sigma_A^2(1-\kappa^2)^2)h) > 1$
- LL Result does not hold

IV. Multiplicative Returns

Case (ii): the CAPM holds in multiplicative returns for all h, then

$$\beta_I = \frac{\exp((\mu_A + .5\sigma_A^2)h) - 1}{\exp((\mu_M + .5\sigma_M^2)h) - 1} \quad (19)$$

Then

$$Beta(A, M, h) = \frac{(\beta_I + (1-\beta_I)\exp(-\mu_M h - .5\sigma^2{}_M h)(\exp(\beta_I \sigma^2{}_M h) - 1)}{(\exp(\sigma_M^2 h) - 1)} \quad (20)$$

- In Case (ii), we have the Levhari-Levi Result

V. Some Empirical Findings

- Sample Data: 50 randomly selected stocks from S&P500 index from July 1, 2009 to July 1, 2010

- Betas we wish to calculate are conditional but we note that the formula do not depend upon initial values (See Equation (10) of Slide 20)

- We estimate daily, weekly and monthly betas based on one year of data, this is a snap shot one could do this on a rolling basis. However it is the same data we use for parameter estimation of the OU model

V. Some Empirical Findings

- It turns out it is easier to estimate de-trended log prices than returns This is because we have an exact discrete linear relationship

$$\begin{bmatrix} q_A(t+h) \\ q_M(t+h) \end{bmatrix} = \begin{pmatrix} e^{-\theta_{A1}h} & \frac{\theta_{A2}}{\theta_{A1}-\theta_M}\left(e^{-\theta_M h} - e^{-\theta_{A1}h}\right) \\ 0 & e^{-\theta_M h} \end{pmatrix} \cdot \begin{bmatrix} q_A(t) \\ q_M(t) \end{bmatrix} + \begin{bmatrix} u_A(t) \\ u_M(t) \end{bmatrix} \quad (A5)$$

- Estimate θ_{A1}, θ_{A2}, θ_M, σ_A, σ_M and κ using a version of generalized least squares (GLS) which will be equivalent to maximum likelihood asymptotically under assumptions of stationarity, then compute exact, sample and instantaneous beta and CV.

V. Some Empirical Findings

- 24 out of 50 of the random sample of stocks have $\theta_{A1} > \theta_M$
- All 50 of θ_M and θ_{A1} were significantly different from 0 at 5% confidence level while only 3 of θ_{A2} were. 6 of θ_{A2} were significant at 10%
- 20 of the beta plots are increasing and 30 of the plots are decreasing

V. Some Empirical Findings

- 16 of the empirical betas are monotonic, this might be thought of as a rejection of the theory. However monotonic in population does not imply monotonic in sample

- In any case, we see this as a useful model rather than a true description of the unknowable reality (We are positivists!)

V. Some Empirical Findings

Table I: Number of Sample Stocks in Different Domain

T1 / T2	$\theta_M > \theta_{A1}$	$\theta_M < \theta_{A1}$
$\beta_I < CV$	3	7
$\beta_I > CV$	23	17

V. Some Empirical Findings

Table II: Empirical Result of 50 Random Samples

Parameters									
	thetaM	thetaA1	thetaA2	sigA	sigM	Kappa	Inst Beta	O(h) CV	
Average	0.029	0.027	0.008	0.009	0.005	0.685	1.126	0.524	
Standard Dev	0.012	0.015	0.016	0.004	0.000	0.140	0.459	5.120	
Betas	**Exact betas**			**Approximated betas**			**Empirical betas**		
	betaD	betaW	betaM	betaD	betaW	betaM	betaD	betaW	betaM
Average	1.127	1.132	1.193	1.122	1.108	1.042	1.132	1.147	1.057
Standard Dev	0.461	0.473	0.656	0.463	0.484	0.649	0.463	0.526	0.460
% Difference	**Approximated % Diff**			**Empirical betas**					
	betaD	betaW	betaM	betaD	betaW	betaM			
Average	0.63%	3.22%	15.76%	-0.52%	-0.44%	2.43%			
Standard Dev	1.49%	7.53%	38.73%	1.71%	19.12%	38.75%			

V. Some Empirical Findings

- Percentage errors are calculated as (exact beta – alternative beta) / exact beta

- Daily results are extremely good (The data does not know that they are OU), weekly results are less good and monthly results are unconvincing

- The average of CV is less than the average of the instantaneous beta and the average of exact betas are larger than that of the instantaneous beta for all h

- We find that the average of daily beta < average of weekly beta < average of monthly beta

V. Some Empirical Findings

Bekaert and Wang (2010)

Exhibit 4: Inflation Betas over Longer Horizons

	Bonds					Stocks				
	1-Year Horizon	2-Year Horizon	3-Year Horizon	4-Year Horizon	5-Year Horizon	1-Year Horizon	2-Year Horizon	3-Year Horizon	4-Year Horizon	5-Year Horizon
Developed countries	0.28 (0.13)	0.59 (0.15)	0.84 (0.17)	1.04 (0.18)	1.12 (0.19)	-0.25 (0.29)	-0.11 (0.27)	-0.05 (0.25)	0.01 (0.23)	0.12 (0.23)
Emerging countries	0.98 (0.34)	1.63 (0.34)	2.02 (0.52)	1.21 (0.57)	2.11 (0.65)	1.01 (0.07)	1.02 (0.03)	1.03 (0.03)	1.03 (0.03)	1.00 (0.03)
North America	0.27 (0.36)	0.67 (0.21)	1.04 (0.30)	1.13 (0.32)	1.39 (0.45)	-0.42 (0.81)	-0.40 (0.69)	-0.31 (0.59)	-0.25 (0.51)	-0.16 (0.45)
European Union	0.30 (0.15)	0.65 (0.19)	0.94 (0.23)	1.11 (0.23)	1.17 (0.23)	0.27 (0.43)	-0.09 (0.35)	-0.05 (0.32)	-0.10 (0.29)	-0.04 (0.28)

V. Some Empirical Findings

Bekaert and Wang (2010)

Exhibit 5: Unexpected Inflation Betas over Longer Horizons

	Bonds					Stocks				
	1-Year Horizon	2-Year Horizon	3-Year Horizon	4-Year Horizon	5-Year Horizon	1-Year Horizon	2-Year Horizon	3-Year Horizon	4-Year Horizon	5-Year Horizon
Developed countries	-0.58 (0.19)	-0.58 (0.15)	-0.22 (0.13)	0.19 (0.13)	0.36 (0.13)	-0.44 (0.40)	-0.59 (0.32)	-0.59 (0.33)	-0.66 (0.34)	-0.58 (0.31)
Emerging countries	0.92 (0.33)	1.57 (0.37)	1.93 (0.54)	1.08 (0.58)	2.09 (0.68)	0.97 (0.09)	0.98 (0.05)	1.03 (0.03)	1.04 (0.03)	1.03 (0.04)
North America	-0.44 (0.48)	-0.62 (0.33)	0.07 (0.19)	0.26 (0.31)	0.47 (0.34)	-0.99 (1.26)	-1.08 (0.90)	-0.74 (0.69)	-0.75 (0.55)	-0.84 (0.41)
European Union	-0.53 (0.19)	-0.43 (0.17)	-0.09 (0.17)	0.23 (0.17)	0.44 (0.16)	-0.24 (0.63)	-0.92 (0.54)	-1.26 (0.43)	-1.39 (0.36)	-1.21 (0.30)

VI. Conclusion

- In this paper we have advocated the use of the OU model in modelling the intervalling effect to give us deeper insight into the behavior of monotonicity, we do not believe it is the true process governing all returns. However over short horizons it seems extremely good when compared with standard estimators
- Allows us to generalise the LL hypothesis to situations where both asset and market are correlated
- It takes the discussion away from whether stocks are aggressive or defensive. Applicable to any one factor

VI. Conclusion

- The generalisation of the LL hypothesis is based on exact calculation. We can also approximate the results using small – h asymptotics, we can easily deal with both additive and multiplicative returns

- We also show that beta will be monotonic in h

- This creates a point which we call the critical value below which beta is underestimated (against the instantaneous beta) as h increases but the bias is decreasing whilst above this point the beta is overestimated as h increases

VI. Conclusion

Further work in this area is planned

- Issues of overlapping data; building daily models based on monthly or annual returns
- Addressing issues of synchronicity with existing model, we can look at deterministic nonsynchronicity such as different market closing time for same asset
- To deal with stochastic synchronicity, we would need to extend our model by randomizing the arrival time of the data

The End