Financial Averaging

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For NORTHFIELD's 24th ANNUAL RESEARCH CONFERENCE, 2011

Averaging is ubiquitous in Finance

- **Expected value** = Probability-weighted AVERAGE
- **DJ Industrial AVERAGE** = Capitalization-weighted AVERAGE
- **Annualization** = AVERAGE yearly value
- **Internal Rate of Return** = Time-weighted AVERAGE return
- **Modified Dietz** = Money-weighted AVERAGE return

?

- What do arithmetic and geometric averages have in common?

- What is the general definition of an average that fits them both?

A Requirement For Any Type Of Average that captures its intuitive meaning.

- The average of a list of identical elements must be that identical element.

- Average of {50, 50} = 50

Simple standard ANNUALIZATION

$R_A \equiv \text{Ann}\{R_t\}$

$1+R_A = [\Pi_{t\epsilon T} (1 + R_t)]^{1yr/T}$ (Why?)

$T = \Sigma_{t\epsilon T} \Delta t.$

Annualization examples

$t_1 = 1$ $t_2 = 1$ $T = t_1 + t_2 = 2.$

|-------------|-------------|

R_1	R_2
1/2	0
0	-1/3
0	-1/2

$1+R_A \equiv 1 + Ann\{R_t\} = [(1+R_1)*(1+R_2)]^{[1yr/(t1+t2)]}$

$1+R_A = 1 + Ann\{1/2, 0\} = [(1 + 1/2)*(1 + 0)]^{1/2} = sqrt(3/2) = 1 + 22.47\%$

$1+R_A = 1 + Ann\{0, -1/3\} = [(1 + 0)*(1 – 1/3)]^{1/2} = sqrt(2/3) = 1 – 18.35\%$

$1+R_A = 1 + Ann\{0, -1/2\} = [(1 + 0)*(1 – 1/2)]^{1/2} = sqrt(1/2) = 1 – 29.29\%$

Standard Annualization of a Geometric Difference of Return Factors

$\Delta^g_t \equiv (1 + R^P_t) / (1 + R^B_t) - 1.$

The Ann of the Geo diff is the Geo diff of the ANN?

$1 + \Delta^g_A \equiv 1 + \mathrm{Ann}\{\Delta^g_{t \varepsilon T}\}$

$= [1 + \mathrm{Ann}\{R^P_{t \varepsilon T}\}] / [1 + \mathrm{Ann}\{R^B_{t \varepsilon T}\}]$ (Why?)

$= \{ \Pi_{t \varepsilon T} [(1 + R^P_t) / (1 + R^B_t)]\}^{1yr/T} = \Pi_{t \varepsilon T} (1 + \Delta^g_t)$

Ex. Ann of a Geo Diff

$t_1 = 1$ $t_2 = 1$ $T = t1 + t2 = 2.$

|---------------|----------------|

$R^P_1 = 1/2$ $R^P_2 = 0$

$R^B_1 = 0$ $R^B_2 = -1/3$

$\Delta^g_2 = 1/2$ $\Delta^g_2 = 1/2$ $\Delta^g_t \equiv (1 + R^P_t) / (1 + R^B_t) - 1.$

$$1 + \Delta^g_A = \{ [(1 + R^P_1)/(1 + R^B_1)] * [(1 + R^P_2) / (1 + R^B_2)] \}^{1yr/T}$$

$$1 + \Delta^g_A = \{ [(1 + 1/2)/(1 + 0)] * [(1 + 0)/(1 - 1/3)] \}^{1/2} = 3/2$$

$\Delta^g_A = Ann^g\{1/2, 1/2\} = 1/2$ (OK)

Standard Annualization of an Arithmetic Difference of Return Factors

$\Delta^a{}_t \equiv [1 + R^P{}_t] - [1 + R^B{}_t] = R^P{}_t - R^B{}_t.$

The Ann of the Arith diff is the Arith diff of the ANN?

$\Delta^a{}_A \equiv \mathrm{Ann}\{\Delta^a{}_{t\varepsilon T}\} \quad = \quad \mathrm{Ann}\{R^P{}_t - R^B{}_t\}$

$? \quad = \mathrm{Ann}\{R^P{}_T\} - \mathrm{Ann}\{R^B{}_T\}$ (Why??)

$= \{ \Pi_{t\varepsilon T} [(1 + R^P{}_t)]\}^{1yr/T} - \{ \Pi_{t\varepsilon T} [(1 + R^B{}_t)]\}^{1yr/T}$

Ex. Ann of an Arith Diff. A Problem!

$t_1 = 1$ $\quad$ $t_2 = 1$ $\qquad$ $T = t1 + t2 = 2.$

|---------------|----------------|

$R^P_1 = 1/2$	$R^P_2 = 0$
$R^B_1 = 0$	$R^B_2 = -1/2$
$\Delta^a_2 = 1/2$	$\Delta^a_2 = 1/2$

$\Delta^a_t = R^P_t - R^B_t.$

$\Delta^a_A \equiv Ann\{\Delta^a_{t\varepsilon T}\} = ? = Ann\{R^P_T\} - Ann\{R^B_T\}$ (Using standard assumption.)

$\Delta^a_A = [(1 + R^P_1) * (1 + R^P_2)]^{1yr/T} - [(1 + R^B_1) * (1 + R^B_2)]^{1yr/T}$

$\Delta^a_A = [(1 + 1/2) * (1 + 0)]^{1/2} - [(1 + 0) * (1 - 1/2)]^{1/2} = Sqrt(3/2) - Sqrt(1/2)$

$\Delta^a_A = Ann^g\{1/2, 1/2\} = 0.518 \neq 1/2.$

So, did this fund out-perform some alternate fund that beat its benchmark by 51% to 0% for each of 2 years?
$\Delta^g_A = \{ [(1 + 0.51) * (1 + 0.51)]^{1/2} - [(1 + 0) * (1 + 0)]^{1/2} \} = 1.510 - 1 = 0.510 = 51\%.$

What is an average?

An intensive property
used to compare values
with different extensions.

Requirements

For any type of average:

1. The average of a list of identical elements must be that identical element.

2. The list can have any finite cardinality.

3. Symmetrical under permutations of the list.

4. $MAX\{X_i\} \geq X_A \geq MIN(X_i\}$. (???)

Formal & General Definition of an Average

An F-average of a list, {…}

is a value that preserves

the F-property of the list

when each value of the list

is replaced by the F-average value.

Requires a choice of F{…}.

Arithmetic Average Preserves the Sum of a List

The arithmetic average of {2,8}
preserves the sum of {2,8}
when the list is changed to {A, A}:

Sum{2, 8} = Sum{A, A}.

2 + 8 = A + A.

Thus, A = (2 + 8)/2 = 5.

Geometric Average Preserves the Product of a List

The geometric average of {2,8}
preserves the product of {2,8}
when the list is changed to {A, A}:

Product{2, 8} = Product{A, A}.

2 * 8 = A * A.

Thus, A = sqrt(2*8) = 4.

General Average Preserves a Function of a List

In general, the F-average, A, of the list $\{X_i\}$ is defined by:

$F(X_1, X_2, \ldots X_N) = F(A, A, \ldots A)$.

To be legitimate, F(…) must be such that

2: Expandable

3: Symmetrical

4: Creates In-Range averages

Annualization of Returns Preserves the Period Return

The Ann R: $R_A \equiv \mathrm{Ann}\{R_t\}$ is defined by:

$R_T\{R_{t\varepsilon T}\} = R_T\{R_A\}$

$\Pi_{t\varepsilon T}\ (1 + R_t) = \Pi_{t\varepsilon T}\ (1 + \mathbf{R_A})$

$1{+}R_A = [\Pi_{t\varepsilon T}\ (1 + R_t)]^{1yr/T}$ (Explained!)

$T = \Sigma_{t\varepsilon T}\ \Delta t.$

Annualization of a Geometric Difference

The Ann Geo Diff of Rs
Preserves the Geometric difference of the whole period, T:

$1 + \Delta^g_T \equiv (1 + R^P_T) / (1 + R^B_T) = [\ \Pi_{t\varepsilon T}\ (1 + R^P_t)\ ] / [\ \Pi_{t\varepsilon T}\ (1 + R^B_t)\ ]$

$1 + \Delta^g_T = \Pi_{t\varepsilon T}\ [(1 + R^P_t) / (1 + R^B_t)] = \Pi_{t\varepsilon T}\ [\ 1 + \Delta^g_t\ ].$

Thus, $\Pi_{t\varepsilon T}\ [\ 1 + \Delta^g_t\ ] = \Pi_{t\varepsilon T}\ [\ 1 + \boldsymbol{\Delta^g_A}\ ],$ where Δ^g_A replaces all Δ^g_t.

$1 + \Delta^g_A = \{\ \Pi_{t\varepsilon T}\ [(1 + R^P_t) / (1 + R^B_t)]\}^{1yr/T}$ (Explained!)

Annualization of an Arithmetic Difference of Returns

$\Delta^a_t \equiv R^P_t - R^B_t$

The <u>Ann Arith</u> Diff preserves the Arith Diff of the whole period, T,
(Here just shown for yearly returns R_t):

$\Delta^a_T \equiv R^P_T - R^B_T = \Pi_{t\varepsilon T}(1 + R^P_t) - \Pi_{t\varepsilon T}(1 + R^B_t)$

$\Delta^a_T \equiv R^P_T - R^B_T = \Pi_{t\varepsilon T}(1 + R^B_t + \Delta^a_t) - \Pi_{t\varepsilon T}(1 + R^B_t)$

$\Pi_{t\varepsilon T}(1 + R^B_t + \Delta^a_t) - \Pi_{t\varepsilon T}(1 + R^B_t) = \Pi_{t\varepsilon T}(1 + R^B_t + \boldsymbol{\Delta^a_A}) - \Pi_{t\varepsilon T}(1 + R^B_t)$

Thus, $\Pi_{t\varepsilon T}(1 + R^B_t + \Delta^a_A) = \Pi_{t\varepsilon T}(1 + R^P_t)$ (Corrected!)

$\Delta^a_A \neq [\Pi_{t\varepsilon T}(1 + R^P_t)]^{1yr/T} - [\Pi_{t\varepsilon T}(1 + R^B_t)]^{1yr/T}$ (50% v. 51.8% Explained!)

Fractional Years

$$\Pi_{t \varepsilon T} \left[(1 + R^{B}{}_{t})^{T/t} + \boldsymbol{\Delta}^{\mathbf{a}}{}_{\mathbf{A}})\right]^{t/T} = \Pi_{t \varepsilon T} (1 + R^{P}{}_{t})$$

Weighted Average

The value that preserves the weighted sum:

$\Sigma_i W_i = 1$

$\Sigma_i W_i * X_i \equiv \Sigma_i W_i * X_A = X_A * \Sigma_i W_i = X_A.$

Note: W_i stays with X_i under permutation.

Negative Weights

Whenever some W_i are outside the range $1 \geq W_i \geq 0$,
There is the possibility for some $\{X_i\}$ that all
X_A are outside range $MAX\{X_i\} \geq X_A \geq MIN(X_i\}$,
And, thus, that all solutions are 'extraneous.'

Ex.

$W_1 * X_1 + W_2 * X_2 = W_1 * X_A + W_2 * X_A$

$1.4 * 0 + (-0.4) * 3 = 1.4 * X_A + (-0.4) * X_A$

$-1.2 = 1 * X_A$

Thus, X_A is OUTSIDE range $3 \geq X_A \geq 0$. Anomalous!

Note Re Shorts

- It can be shown that negative weights in general create these types of problems for financial calculations,
- And, thus, that shorts need to be dealt with by models other than those that assign them negative weights,
- e.g. in B&H Ret, mD, IRR and for other 'avg'.

Preserved Function for IRR
('MWR' in presence of mid-period trades)

State the final value of an investment
(e.g. issue, component or fund)
As a function of its annualized return factor ($\mathcal{F}$) <u>to close</u>
When there are Cash Flows
(i.e. purchases and sales to and from an external source):

$$V_f = V_o * \mathcal{F}_o^{T/yr} + \Sigma_{t \varepsilon T} [CF_t * \mathcal{F}_t^{(T-t)/yr}]$$

Where $R_t = \mathcal{F}_t - 1$ is the annualized return from t to T
And CF_t is the CF at time t after the open.

IRR

The average of $\mathcal{F}_t$ that preserves this V_f function is defined:

$$V_o * \mathcal{F}_o^{T/yr} + \Sigma_{t\varepsilon T} [CF_t * \mathcal{F}_t^{(T-t)/yr}] = Vo * \boldsymbol{\mathcal{F}}_{\mathbf{A}}^{T/yr} + \Sigma_{t\varepsilon T} [CF_t * \boldsymbol{\mathcal{F}}_{\mathbf{A}}^{(T-t)/yr}].$$

This is standard definition of the return factor for the Internal Rate of Return (IRR).

This explication makes explicit that

IRR is the CF-weighted average that preserves V_f of the annualized return-till-end of the specified period.

Preserved function for the Modified Dietz Return ('MWR' in presence of mid-period trades)

Consider the final value, V_f, of an instrument as a function of its return per proportion of time till the end of a specified period, R^{md}_t.

$$R^{md}_t \equiv R_{t\to T} /[(T-t)/T] \qquad G \equiv V_f - (V_o + \Sigma_{t\varepsilon T}\, CF_t\,)$$

$$V_f = V_o * (1 + R_{o\to T}) + \Sigma_{t\varepsilon T}\, [CF_t * (1 + R_{t\to T}\,)]$$
$$= V_o * (1 + R^{md}_o) + \Sigma_{t\varepsilon T}\, [CF_t *(1 + R^{md}_t * (T\text{-}t)/T)].$$
$$G \;= V_o * R^{md}_o + \Sigma_{t\varepsilon T}\, [CF_t * \; R^{md}_t * (T\text{-}t)/T].$$

Modified Dietz Return

The average of R^{md}_t that preserves this Gain function is defined by:

$V_o * R^{md}_o + \Sigma_{t\varepsilon T} [CF_t * R^{md}_t * (T\text{-}t)/T] \equiv V_o * \mathbf{R^{md}} + \Sigma_{t\varepsilon T} [CF_t * \mathbf{R^{md}} * (T\text{-}t)/T]$.

Thus, $R^{md} = G/\{ V_o + \Sigma_{t\varepsilon T} [CF_t * (T-t)/T] \}$,

Which is the standard definition of the Modified Dietz Return.

Thus, while R^{md} is usually thought of as
Modified Dietz = Gain/(Time-weighted average of the CFs)
It more meaningfully is the CF-weighted average of
the return till period-end per proportion of time till period-end
that preserves the Gain (&, thus, V_f). (Relevance?)

Averaging Summary

Clarifying the general definition of Average:

- Provides a justification for many standard financial averaging procedures, Such as Arith, Geo, Ann{R_t} and $\Delta^g R_A$.
- Clarifies the meaning & implications of others Such as IRR, R^{md} and short weights, &
- Corrects the calculation of some of them, Such as $\Delta^a R_A$.

Advanced Averaging

- Fractional years for active returns.
- Annualization of Correlations, SDs
  and other risk parameters.
- Annualization of decision attributes.

Warning!

If any average values,

Including annualizations,

Are employed in any situation

then special care is necessary

Regarding the calculations of these averages

If they are not to be misleading

in all but the very simplest of cases.