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Portfolio Optimization with VaR, CVaR, Skew and Kurtosis

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Why is this Issue Important?

- Since the theoretical advent of mean-variance, portfolio optimization in the 1950s there has been an ongoing debate as to the necessity of including higher moments of return distributions (skew and kurtosis) into the process.
- In recent years, the increasing regulatory focus on downside “tail risk” measures such as VaR and CVaR has extended the interest in this topic.
- The much of the literature on this issue unfortunately ignores some of the important real world elements of the problem such as transaction costs and parameter estimation error

Presentation Outline

- This presentation will first identify the portfolio situations where (and where not) the importance of higher moments appears to be economically material and statistically significant in optimization

- We will illustrate two broad approaches to incorporate higher moments into portfolio optimization.
  - Our first approach will use “full scale” optimization that explicitly includes skew and kurtosis in the objective function.
  - In the second approach, we will consider using analytical techniques to reduce the four-moment problem to an comparable mean-variance problem, before solving conventionally.

- Due to the estimation error in the parameters, we find transforming the problem into mean variance equivalence to be sufficient for all but the most extreme cases.

VaR and CVaR

- Value at Risk and Conditional Value at Risk are measures of probable loss that are popular with financial intermediaries (i.e. banks)
  - Value at Risk is defined as value X such that my loss in (100-P%) of events will be greater than X
  - Conditional Value at Risk is defined as the expected value C of the loss in the (100-P%) of events when the loss is greater than X

- If asset returns are normally distributed, VaR and CVaR are simple scalars of volatility
  - There is nothing special to do in terms of mean-variance optimization

VaR and CVaR for Complex Cases

- If the distribution of asset returns is continuous and differentiable, VaR and CVaR are functions of our usual four moments (mean, SD, skew and kurtosis)
  - Many empirical studies suggest daily financial returns fit a T-5 distribution
- Alternative empirical studies have suggested that a stable paretian distribution better fits high frequency return data
  - These distributions have an undefined second moment
- Distributions for some complex financial instruments must be estimated by numerical simulation and then a distributional assumption can usually be made

Higher Moments When Should We Care

- Markowitz (1952) says he would of preferred to use semi-variance as the risk measure instead of variance
  - Too computationally difficult for the 1950s
  - He later argued that just mean and variance were sufficient for practical cases

- Samuelson (1970) argues that investors should define their choices to maximize utility over all moments

- Hlawitscka and Stern (1995) show the simulated performance of mean variance portfolios is nearly indistinguishable from the utility maximizing portfolio

More on Higher Moments

- Wilcox (2000) shows that the importance of higher moments is an increasing function of investor gearing
- Cremers, Kritzman and Paige (2003)
  - Use extensive simulations to measure the loss of utility associated with ignoring higher moments in portfolio construction
  - They find that the loss of utility is negligible except for the special cases of concentrated portfolios or "kinked" utility functions (i.e. when there is risk of non-survival).
- Satchell (2004)
  - Describes the diversification of skew and kurtosis
  - Illustrates that plausible utility functions will favor positive skew and dislike kurtosis

Optimization with Higher Moments

- Chamberlin, Cheung and Kwan(1990) derive portfolio optimality for multi-factor models under stable paretian assumptions

- Lai (1991) derives portfolio selection based on skewness

- Davis (1995) derives optimal portfolios under the Gamma distribution assumption
  - Consistent with Gulko (1997)

- Papers with examples four moment optimization
  - Harvey (2003)
  - Kemalbay, Ozut, Franko (2011)
  - Beardsley, Field, Xiao (2012)
  - Saranya and Prasanna (2014)
  - Harvey, Liechty, Liechty and Muller (2013)

Why Not Just Use Four Moment Optimization?

- Higher moments of distributions come from rare events
  - Estimation error on higher moment is large from empirical data
  - When you are wrong on the sign of skew you are very wrong

- Think about a standard error of the correlation of two assets
  - $SE(p) = (1 - p^2) / (n-2)^{.5}$

Lets work through the arithmetic for p = .3 and N = 120, and N = 60

Four Moment Optimization is Hard to Do

- The four moment objective function looks like for a "growth optimal" investor

$$U = A - (.5 * S^2) + (2/3) * MS_m^3 - (1/2) * KS_m^4$$

S = volatility

M = skew

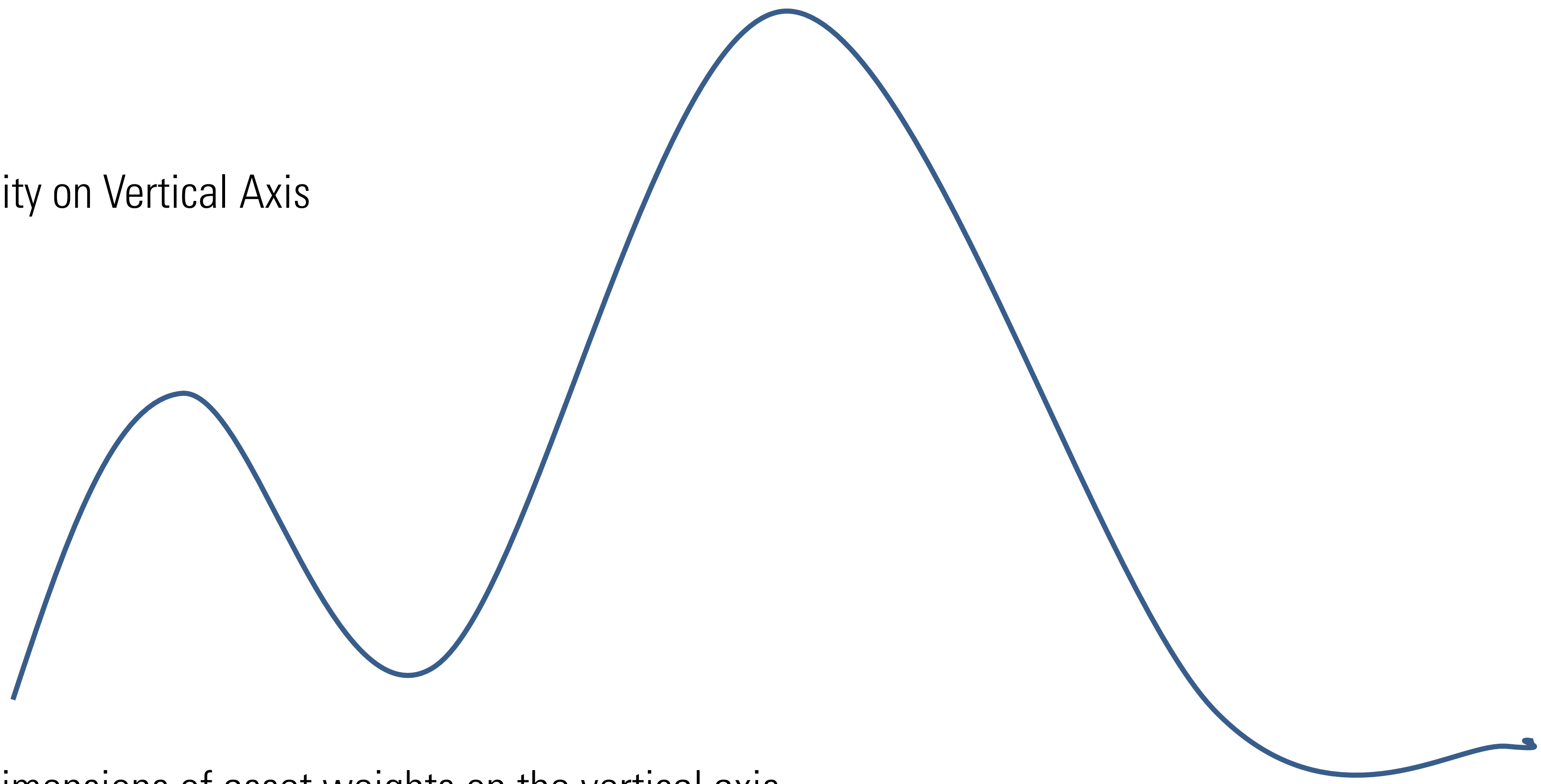
K = excess kurtosis

A = return

Since M and K can have either positive or negative values this problem is not a convex polynomial. There is no guarantee of a single global maximum

My Bad Drawing of the Non-Convex Issue

Utility on Vertical Axis



N dimensions of asset weights on the vertical axis

Dealing with Non Convex Optimization

- The usual solution for optimizing difficult functions is to invoke Metropolis Hastings algorithm
  - AKA Markov Chain Monte Carlo simulations
- What you are randomizing is the *initial* portfolio
- Do hundreds of optimizations with different starting points
  - If they all end up in the same solution you are pretty confident that you have global maximum
  - Very computationally intensive if you have lots of assets
- Randomizing the initial portfolio is problematic if you have transaction costs that are material

Convert the Four Moment Problem to Just Two

- Cornish-Fisher (1937)
  - We can convert any four moment distribution into an approximately equivalent two moment distribution
  - We adjust the mean and standard deviation to best fit the desired percentile (the P from the VaR/CVaR definitions) for the subject distribution

- We can also estimated an adjusted volatility V*

$$V^* \sim (S^2 - (200/RAP)\,[(2/3)^*\ MS^3 + (1/2)^*\ KS^4])^{.5}$$

In the absence of explicitly stated risk aversion we approximate RAP = 6 * S

Signed Constrained Weights

- If the asset weights in the optimization are sign constrained, the transformation to mean-variance is easier

- Consider a call option on a stock
  - The payoff distribution of a call option has skew
  - If you are long the option the skew is positive
  - If you are short the option the skew is negative
  - For any selected percentile (P) I can estimate the volatility only from the lower tail of the distribution

- Similar to estimation of a semi-volatility around a target returns (e.g. Rom/Sortino 1983)

A Primer on Catastrophe Bonds

- Our basic framework is to consider a catastrophe bond as a portfolio which is long a Treasury bond and short a lottery ticket (i.e. a weather lottery).
  - We represent the lottery ticket as a synthetic security whose only property is a unit amount of idiosyncratic risk.
  - Given the yield on the CAT bond, and the expected loss from the term sheet (typically 2%), we can back into the notional value of the lottery ticket that gives the correct absolute volatility for the CAT bond.
  - If you own more than one CAT bond such that the lottery risks are correlated (all Florida hurricane risk), you can define the same synthetic security as part of multiple CAT bonds, thereby creating a correlation effect among the lottery payouts.
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Catastrophe Bond

- If a catastrophe bond has a 2% chance of losing everything,
  - There is a 98% probability that the volatility will be equivalent to the related Treasury bond with a 2% likelihood of 100% loss
  - Calculate the standard deviation, skew and kurtosis associated with the lottery ticket(s) from a binomial distribution
  - Add in the volatility of the Treasury bond without covariance
  - Convert to a two moment distribution as described
- Fun Facts
  - 95% VaR (the most popular P value) is zero, since 95% of the time your loss due to catastrophe will be zero
  - 95% CVaR is 40%

Catastrophe Bond Portfolio Optimization

- Run a four moment optimization on a portfolio of N independent catastrophe bonds
  - Return estimate is the yield to maturity minus the 2% loss expectation
  - Obtain the efficient frontier of optimal portfolios
- Convert the return distribution of each bond to two moments using just the volatility adjustment
  - No change in return estimate
  - Obtain the efficient frontier of optimal portfolios
- The two efficient frontiers are not statistically significantly different under the test from Jobson (1991) until N is very small

Conclusions

- A consensus of research suggests that optimizing inclusive of higher moments (or VaR, CVaR) is really useful only when the investor has non-quadratic utility

- Focus on empirically obtained estimates of higher moments worsens estimation error as you are optimizing on evidence of rare events with based on small samples

- Full scale optimization (four moment) does not produce materially different results from “adjusted” mean-variance for the vast majority practical cases