An Integrated Multi-Period Approach to Risk and Asset-Liability Management

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The Great Divide

**Risk Management**
- Is primarily concerned with the uncertainty of the value of assets, and its implication to asset allocation and portfolio construction.

**Asset-Liability Management**
- Its fundamental objective is to assure the availability of funds at points in time in the future when they are needed to cover exogenous divestment.

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The Ways of Risk Management

- What is the volatility and correlation among assets
- What is the price of risk and return thereof
- How can we optimize the tradeoff between return and volatility/correlation
- MPT found typical solutions in linear models and MVO or related techniques
- It deals in the “Single Period” environment – a month to a year
The Ways of Asset-Liability Management

- Tries to match cash inflows and outflows in the future; difficult to match all
- Looks for approximate simultaneous measures of cash flow matching – i.e. duration
- May wish to relax the deterministic view of cash flows and simulate scenarios
- Hard to optimize asset mix vs. exogenous liabilities through simulation alone, so...
- Abandons the “matching” objective and only optimizes for A – L net present values (deficit / surplus)
ALM: The Problem With Duration Matching

- Let’s view a long term liability as an annuity (bond) with duration D
- We seek out a bond asset that has the same duration D
  - That may be a zero-coupon bond with a maturity of now + D years
  - Or it can be a coupon bond with 100 year maturity and appropriately chosen coupon
  - Or it can be a coupon bond with any other maturity and appropriately chosen coupon
- No clear match at all because:
  - Duration builds on PV which is indifferent between timing of two cash flows as long as their discounted value is the same
  - We are certainly not indifferent if we have the required funds to cover a liability cash flow L at time T, vs. T1
The Question of Duration

Waldorf: The question is....What is a “Duration”
Statler: The question is ... Who Cares
ALM: Problems with Surplus Measures

Surplus / Deficit abandon all notion of timing of future cash inflows and outflows

It poses the thorny question of what discount rate to use for the liabilities

- this is a theoretically wrong question to ask
- opens the door for an array of extraneous factors (e.g. a single QE move wipes the surplus); is this a worry about the pensioners future funds, or about the image of the “agents”.
- creates skewed objectives and yield-frenzy.
• **Dr. Bob**: Nurse, the patient has reached a critical condition. He needs to be taken **immediately** to see a **DOCTOR**
Should Risk and ALM be distinct functions?

• Isn’t the objective of Risk Management to maximize future utility of investors?

• Paying mandatory future cash flows at particular times are part of investors’ future utility

• The staging of mandatory cash flows over times makes a single period risk management too restrictive

• Can we adapt our current practices to take care of the time objective.
Waldorf: Well, you gotta give them credit.
Statler: Why's that?
Waldorf: No matter how many times they fail, they will keep doing the same, trying to get things right.
The Proposed Approach

We propose a method to address all relevant issues in a stage-wise manner, but first we stop our attention on:

- The multi-period expected compound return of assets derived from the single period arithmetic mean return
- Derivation of the variance of the multi-period compound return
- Observing the character of actuarial liabilities
- Adapting a stage-wise asset compound process to include the impact of liabilities
Multi-Period Compounded Expected Return

An usual assumption is that returns over different non-overlapping periods of time in the future are I.I.D.

That would make $1 + \text{the return in period } t_1$ independent from the $1 + \text{the return in period } t_2$

That would mean:

$$E(1 + r_{t1})(1 + r_{t2}) = E(1 + r_{t1})E(1 + r_{t2})$$

If we don’t have a view on differences of $R$ over the different periods, then our expression becomes

**Expected Compounded Return** $ECR = (1 + \mu)^k - 1$
Variance of Compounded Return (VCR)

We derive the following formula for the variance of assets of periodically compounded return over time T consisting of K periods:

\[ VCR = \left[ \sigma^2 + (1 + \mu)^2 \right]^k - (1 + \mu)^{2k} \]  \hspace{1cm} (1)

Where “\( \sigma \)-squared” is the single period return variance and “\( \mu \)” is the single period expected return.

COROLLARY: If we consider the asset portfolio return stationary, once we derive the single period mean return and volatility we can project asset performance to any number of periods K.
Derivation of VCR

\[ R_C \] – compound return over time \( T \), \( p_i \) is the probability for a return path, \( n \) is the number of paths, and \( m \) is the overall number of periods, and \( r_k \) is the periodic return in period \( k \)

\[
VCR = E(R_C - E(R_C))^2 = E(R_C^2) - [E(R_C)]^2
\]  \hspace{1cm} (2)

\[
\sum_{i=1}^{n} p_i \left[ \prod_{k=1}^{m} (1 + r_k) - 1 \right]^2
\]

\[
= \sum_{i=1}^{n} p_i \left[ \prod_{k=1}^{m} (1 + r_k)^2 - 2 \prod_{k=1}^{m} (1 + r_k) + 1 \right]
\]

Since \( r_k \) are independent, so are the \((1 + r_k)^2\) terms, so ...

\[
= \sum_{i=1}^{n} p_i \prod_{k=1}^{m} (1 + r_k)^2 - 2 \sum_{i=1}^{n} p_i \prod_{k=1}^{m} (1 + r_k) + 1
\]
Derivation of VCR (cont’d)

\[ VCR = \prod_{k=1}^{m} E[(1 + r_k)^2] - 2 \prod_{k=1}^{m} E(1 + r_k) + 1 \]

\[ = \prod_{k=1}^{m} [\sigma_k^2 + (1 + \mu_k)^2] - 2 \prod_{k=1}^{m} (1 + \mu_k) + 1 \]

In the stationary case:

\[ VCR = [\sigma^2 + (1 + \mu)^2]^k - 2(1 + \mu)^k + 1 - [(1 + \mu)^k - 1]^2 \]

\[ = [\sigma^2 + (1 + \mu)^2]^k - 2(1 + \mu)^k + 1 - (1 + \mu)^{2k} + 2(1 + \mu)^k - 1 \]

\[ = [\sigma^2 + (1 + \mu)^2]^k - (1 + \mu)^{2k} \]
Modeling Actuarial Liabilities

Social Security
Official Social Security Website

Actuarial Life Table

Office of the Chief Actuary
Life Tables

A period life table is a statistical tool used to analyze the mortality experience of a population. For this table, the number of years exposed is calculated using the mortality rates listed below.

<table>
<thead>
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<th>Exact age</th>
<th>Death probability</th>
<th>Number of lives</th>
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<td>0.000161</td>
<td>99,186</td>
</tr>
<tr>
<td>6</td>
<td>0.000150</td>
<td>99,170</td>
</tr>
</tbody>
</table>
Modeling Actuarial Liabilities (cont’d)

- The number of non-renewal events at any time $t$ sum to be $N \sim (\mu_L, \sigma_{L_t}^2)$
- Given a uniform content of age cohorts of pensioners/insureds the interim $t_n$ to $t_{n+1}$ distributions of non-renewal events become independent
- We construct liability returns of the associated payoffs by measuring them against the **current value of assets**
Bridging the A-L Gap

Eventually, we have determined that over \( k \) periods, a mix of our current assets and projected liabilities will have the following distributional parameters, and assuming asset and liabilities are uncorrelated

\[
ECR_k = (1 + \mu_a)^k - 1 - \mu_{L_k}
\]

\[
VCR_k = [\sigma_a^2 + (1 + \mu_a)^2]^k - (1 + \mu_a)^{2k} + \sigma_{L_k}^2
\]

Over \( k+1 \) periods the above parameters compound to:

\[
ECR_{k+1} = [1 + (1 + \mu_a)^k - 1 - \mu_{L_k}] (1 + \mu_a) - 1 - \mu_{L_{k+1}}
\]

\[
VCR_{k+1} = [VCR_k + ECR_k^2][\sigma_a^2 + (1 + \mu_a)^2] - 2[(1 + \mu_a)^k - 1 - \mu_{L_k}]
\]

\[
[(1 + \mu_a) - \mu_{L_{k+1}}] - ECR_{k+1}^2 + \sigma_{L_{k+1}}^2
\]
Better Measures of Deficit

Given the parameters of the distribution ECR and VCR over time T, we can produce more robust measures of surplus that are geared to horizon T

\[ ECR_{k+1} = (1 + (1 + \mu_a)^k - 1 - \mu_{L_k})(1 + \mu_a) - 1 - \mu_{L_{k+1}} \]
\[ VCR_{k+1} = [VCR_k + ECR_k^2][\sigma_a^2 + (1 + \mu_a)^2] - 2[(1 + \mu_a)^k - 1 - \mu_{L_k}] \]
\[ [(1 + \mu_a) - \mu_{L_{k+1}}] - ECR_{k+1}^2 + \sigma_{L_{k+1}}^2 \]

\[ R_{A-L_T} \sim (ECR_T, VCR_T) \]

Expected A-L Shortfall (EALS) = \[ \int_{-\infty}^{0} R_{A-L_T} f dR \]

Probability of A-L Shortfall (PALS) = \[ \int_{-\infty}^{0} f dR \]
Applications to Pensions and Insurance

Four types of entities that will benefit by the approach:

- Defined Benefit Pension Plans
- Defined Contribution Pension Plans / Asset Owners
- Equity-based Life Insurance Companies
- Mutual Life Insurance Companies
Application to DB Plans

a) Liabilities might need to be adjusted for inflation – this creates possible correlation with assets, but can be captured with interim period beta.

b) The Sponsor has a call option on over-performance (surplus), measured in the naïve way. This option’s exercise region correspond to states of the world spanned by our approach - we can simply adjust the expected return of period \([0,T_1]\), by the probability and expected value of exercising that option at time \(T_1\) (and so for all other times). The variances of the subsequent A/L periodic distributions don’t get impacted.

c) The Plan has a put option on under performance (deficit), measured in the naïve way. The compound expected value of period \([0,T_N]\) can be adjusted in the same way as in point b.
Application to DC Plans and Asset Owners

There is no actuarial commitment on the side of the plan, or plan sponsor – investors put money in and take money out at will. The long term analysis of the asset side in isolation is entirely applicable.
Application to Stock Insurance Companies

- The above framework applies completely, with the exception that the surplus call option is not applicable here.

- Insurance companies have premiums coming in which are akin counter-liabilities occurring based on the same probabilities identified before and can be handled in the same framework but with a reverse sign to liabilities.
Application to Mutual Life Companies

• Same as stock insurance companies with the exception that both the surplus call option and deficit put option are not applicable here.

• In some cases periodic payments (premiums) won’t be coming in, depending on the company/policy setup.
What about Optimization

Given the distribution of final surplus at time $T$

$$R_{A-L_T} \sim (ECR_T, VCR_T)$$

We can use the usual exponential utility function:

$$U = -e^{-\frac{1}{\lambda} R}$$

Which would yield an objective function:

$$E(U) = ECR - \frac{1}{2\lambda} VCR$$

Note that expressing liabilities as a drag on asset return over the full time $T$ has given us at least two benefits:

- Does not require us to rescale liability return due to changes in asset values in prior periods
- Gives us grounds to assume constant risk aversion once we have netted out the effect of liabilities (Discretionary Wealth Hypothesis, Wilcox 2003)
What about Optimization (cont’d)

• The ECR and VCR expressions have terms of high order of the independent variables – the asset portfolio weights, so quadratic programming is not an option.

• But the objective of $E(U)$ is eventually a smooth polynomial and it can be solved using numerical algorithms, also incorporating constraints.

• If we don’t want to endure the technical effort of setting up such optimization algorithm, may be we can think of shortcuts.
Optimization (cont’d)

We can rewrite our usual utility function in an periodically expanded way:

\[ U = -e^{\frac{-1}{\lambda}R} = -(e^{\frac{-1}{\lambda}r} \times e^{\frac{-1}{\lambda}r} \times e^{\frac{-1}{\lambda}r} \ldots e^{\frac{-1}{\lambda}r}) \]  

(3)

Where \( r = R / K \)

We can specialize (3) to provide different return realizations over different periods.

\[ U = -(e^{\frac{-1}{\lambda}r_1} \times e^{\frac{-1}{\lambda}r_2} \times e^{\frac{-1}{\lambda}r_2} \ldots e^{\frac{-1}{\lambda}r_k}) \]

This appeals to our intuition because utility compounds, and is still an increasing and concave (risk-averse) function to each periodic return realization.

As long as the multiplicative terms are independent it can be shown that:

\[ E(U) = (e^{\mu_1 \frac{-1}{2\lambda}\sigma_1^2} \times e^{\mu_2 \frac{-1}{2\lambda}\sigma_2^2} \times e^{\mu_3 \frac{-1}{2\lambda}\sigma_3^2} \ldots e^{\mu_k \frac{-1}{2\lambda}\sigma_k^2}) \]

\[ = e^{\mu_1 \frac{-1}{2\lambda}\sigma_1^2+\mu_2 \frac{-1}{2\lambda}\sigma_2^2+\mu_3 \frac{-1}{2\lambda}\sigma_3^2+\ldots+\mu_k \frac{-1}{2\lambda}\sigma_k^2} \]

Where the periodic distributional parameters include those of the liabilities, as calculated form the compound case.
Summary

• A robust framework for incorporating periodic liabilities into long term risk management

• Is mathematically tractable and does not require expensive simulations, and by construction is more accurate as it does not sample points

• Captures long term correlation among assets, and assets and liabilities

• Does not require a liability discount rate

• Since periodic mean is a parameter for VCR and VCR is used in optimization with a negative sign, optimization based on it penalizes estimation error of the mean return – a common problem with MVO; a naturally robust optimization

• Has application far beyond pensions and life insurance; e.g. long term asset management, CAT-bond analysis, Sovereign Credit Risk, etc.
References


