“To Rebalance or Not to Rebalance?”

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To Rebalance or Not to Rebalance

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It is not "the question" but still …

- To rebalance – fixed-weight (FW); Not to – Buy and hold (BH)
- "Passive" (inactive) versus "active"
- Efficient market theory versus market inefficiency
- Traditional cap-weighted indices versus alternative betas
- Asset allocation FW policy versus asset-level BH benchmarks

To Rebalance or Not to Rebalance

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There have been no satisfactory answers

- Does FW portfolios have higher returns?
  - Is "diversification return" real or imaginary?
- Does FW portfolios have lower risks?
- The effects of mean-reverting or trending on portfolio rebalancing
  - FW sells winners and buy losers (in long-only portfolios)
- Effects of portfolio rebalancing for long-short portfolios?
- Should we care more about terminal wealth?

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Outline

- Direct comparison between FW and BH portfolios
- Terminal wealth instead of average returns
- Expected value and variance of terminal wealth – wealth Sharpe ratio
- Long-only portfolios and long-short portfolios
- Effects of serial correlation (a hard problem)
- Qian, Edward, 2014, "To Rebalance or Not to Rebalance: A Statistical Analysis of Terminal Wealth of Fixed-weight and Buy-and-Hold Portfolios", available at www.ssrn.com

Rebalancing Return

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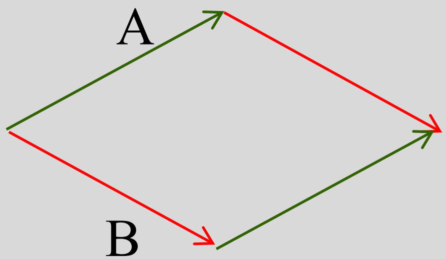
A simple experiment

» **Two securities A and B go up and down with zero cumulative return**



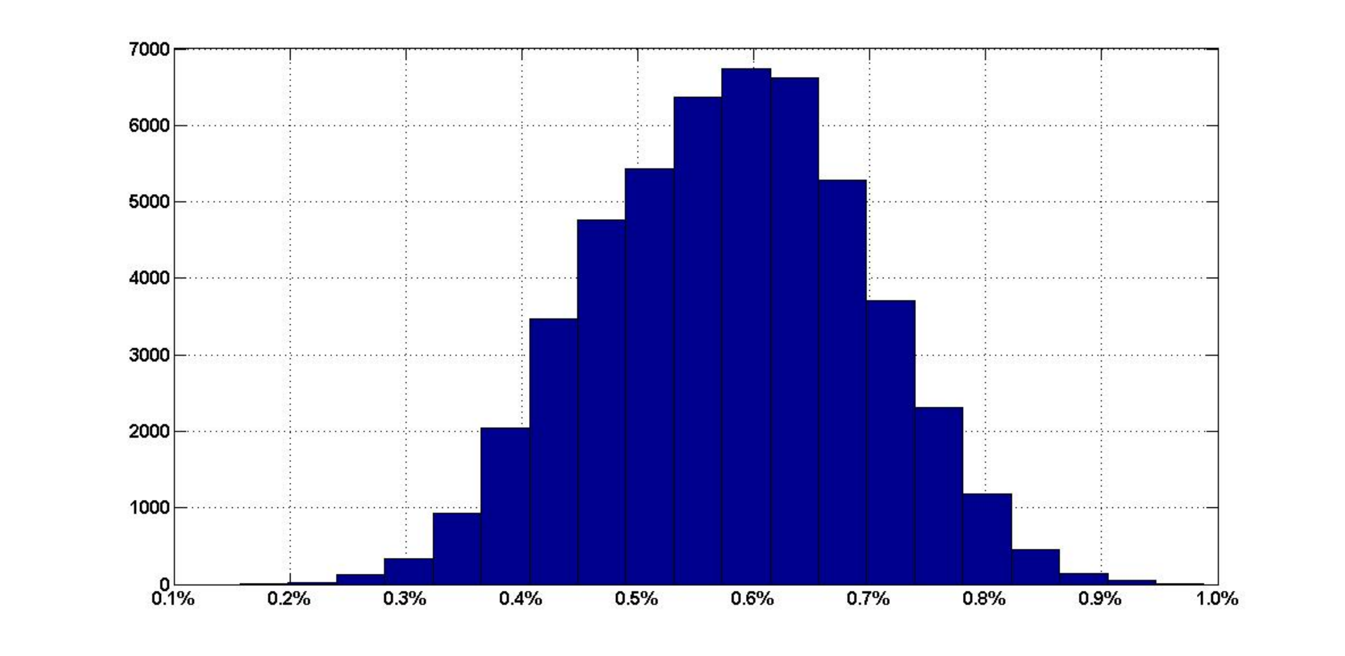
» **Portfolio rebalancing generates positive return**

	Year 1	Rebalance	Year 2
A $50	$100	$62.5 (50%)	$31.25
B $50	$25	$62.5 (50%)	$125
Total $100	**$125 (25%)**	**$125**	**$156.25 (25%)**

Rebalancing Return

A more realistic experiment

- S&P 500 sector portfolios: rebalancing always leads to higher return
  - Annual returns from 1990 – 2013 for 10 S&P sectors
  - 50,000 randomly generated portfolios
  - Alpha = annual return with rebalancing minus return with buy-and-hold



Terminal Wealth

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Notations

» **M assets/N periods, return of ith asset in period n:** r_{in}

» **Expected return independent of n – return vector:** $\vec{\mu}$

$$E\left(r_{in}\right) = \mu_i, i = 1, \cdots, M, n = 1, \cdots, N$$

» **Covariances independent of n – covariance matrix:** Σ

$$E\left[\left(r_{in} - \mu_i\right)\left(r_{jn} - \mu_j\right)\right] = \sigma_{ij}, i, j = 1, \cdots, M, n = 1, \cdots, N$$

» **No serial correlation between returns of different time period**

» **Initial portfolio weights**

$$\vec{w} = \left(w_1, \cdots, w_N\right)'$$

Terminal Wealth

Notations

» **Expected return of the FW portfolio**

$$\mu_p = w_1\mu_1 + w_2\mu_2 + \cdots + w_M\mu_M = \sum_{i=1}^{M} w_i\mu_i = \vec{w}' \cdot \vec{\mu}.$$

» **Volatility of the FW portfolio**

$$\sigma_p^2 = \sum_{i,j=1}^{M} w_i\, w_j\, \sigma_{ij} = \vec{w}'\,\Sigma\vec{w}.$$

Terminal Wealth

Terminal wealth of $1 investment

» **FW portfolio – product of period returns**

$$W_{\mathrm{FW}} = \left(1 + \sum_{i=1}^{M} w_i\, r_{i1}\right) \cdots \left(1 + \sum_{i=1}^{M} w_i\, r_{iN}\right) = \prod_{n=1}^{N} \left(1 + \sum_{i=1}^{M} w_i\, r_{in}\right).$$

» **BH portfolio – weighted sum of terminal wealth**

$$W_{\mathrm{BH}} = w_1 (1 + r_{11}) \cdots (1 + r_{1N}) + \cdots + w_M (1 + r_{M1}) \cdots (1 + r_{MN})$$

$$W_{\mathrm{BH}} = \sum_{i=1}^{M} w_i \left[\prod_{n=1}^{N} (1 + r_{in}) \right].$$

Terminal Wealth

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Expected terminal wealth

» **FW portfolio**

$$E(W_{FW}) = E\left[\left(1 + \sum_{i=1}^{M} w_i\, r_{i1}\right) \cdots \left(1 + \sum_{i=1}^{M} w_i\, r_{iN}\right)\right] = \left(1 + \mu_p\right)^N$$

$$E(W_{FW}) = \left(1 + \sum_{i=1}^{M} w_i \mu_i\right)^N$$

» **BH portfolio**

$$E(W_{BH}) = w_1\, E[(1 + r_{11}) \cdots (1 + r_{1N})] + \cdots + w_M\, E[(1 + r_{M1}) \cdots (1 + r_{MN})]$$

$$E(W_{BH}) = \sum_{i=1}^{M} w_i (1 + \mu_i)^N.$$

Terminal Wealth

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Expected terminal wealth

- **Theorem: for long-only portfolios, i.e., $w_i \geq 0, \sum_{i=1}^{M} w_i = 1$ the expected terminal wealth of the BH portfolio is higher than that of the FW portfolio**

$$\mathrm{E}(W_{\mathrm{BH}}) \geq \mathrm{E}(W_{\mathrm{FW}})$$

- **Proof by Jensen's inequality (convex function)** $\mathrm{f}(\boldsymbol{x}) = (\mathbf{1} + \boldsymbol{x})^{\boldsymbol{N}}$

$$\sum_{i=1}^{M} w_i (1 + \mu_i)^N \geq \left(1 + \sum_{i=1}^{M} w_i \mu_i\right)^N$$

- **Intuition: don't sell winners if winners keep on winning**

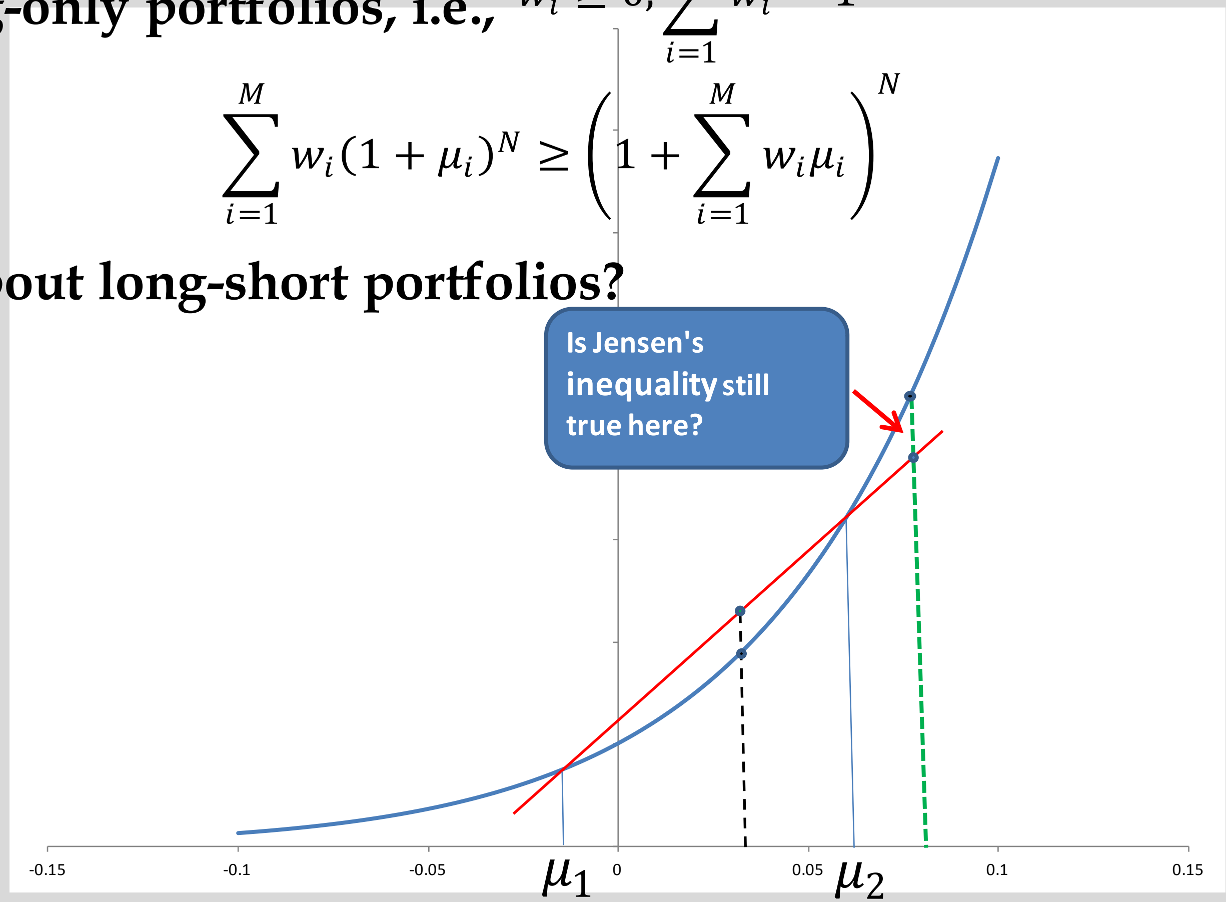
Terminal Wealth

Jensen's inequality

- **For long-only portfolios, i.e.,** $w_i \geq 0, \sum_{i=1}^{M} w_i = 1$

$$\sum_{i=1}^{M} w_i (1+\mu_i)^N \geq \left(1 + \sum_{i=1}^{M} w_i \mu_i\right)^N$$

- **What about long-short portfolios?**



Long-Short Portfolios

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What about long- short portfolios?

- **Short positions: negative weights**
  - **Short selling: borrow shares to sell**
  - **Borrow money to buy assets**
    - Invest with derivatives (futures)
- **Mathematically, we still have** $\sum_{i=1}^{M} w_i = 1$
- **Portfolio leverage if some weights are negative**

$$L = \sum_{i=1}^{M} |w_i| > 1$$

Long-Short Portfolios

Weights of L/S portfolios

- **L/S 120/20 portfolio with security A and B**
  - **A returns 100% and B returns -50%**

	Year 1	Year 1	Rebalance
A($120/120%)	$240	104%	$276 (120%) (Buy)
B(-$20/-20%)	-$10	-4%	-$46 (-20%) (Sell)
Total $100(140%)	**$230**	**100%(108%)**	**$230**

- **Portfolio grows from $100 to $230**
  - **Leverage decreases from 140% to 108%**
  - **Rebalancing leads to <u>releveraging and buying the winner and sell the loser</u>**

Long-Short Portfolios

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Weights of L/S portfolios

- **L/S 120/20 portfolio with security A and B**
  - **A returns -50% and B returns 100%**

	Year 1	Year 1	Rebalance
A($120/120%)	$60	300%	$24(120%) (Sell)
B(-$20/-20%)	-$40	-200%	-$4 (-20%) (Buy)
Total $100(140%)	**$20**	**100%(500%)**	**$20**

- **Portfolio drops from $100 to $20**
  - **Leverage increases from 140% to 500%!**
  - **Rebalancing requires deleveraging and buying the winner and sell the loser**

Long-Short Portfolios

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Weights of L/S portfolios

- When L/S portfolios have gains (losses), leverage decreases (increases)
  - When a L/S portfolio is positioned correctly, i.e., long higher return assets and short lower return assets, its leverage decreases.
  - When a L/S portfolio is positioned wrongly, i.e., long lower return assets and short higher return assets, its leverage increases!
- Buy-and-hold (passive) and leverage don't mix
- FW might perform better than BH

$$\mathrm{E}(W_{\mathrm{BH}}) \leq \mathrm{E}(W_{\mathrm{FW}})$$

Long-Short Portfolios

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Expected terminal wealth

» **Theorem: If** $w_1 < 0$, $w_i \geq 0, i = 2, \cdots, M$. **And** $\sum_{i=1}^{M} w_i = 1$

» **In addition,** $\mu_i \geq \mu_1, i = 2, \cdots, M$, **and** $\mu_p = \sum_{j=1}^{M} w_j \mu_j \geq \mu_i$

» **Then**

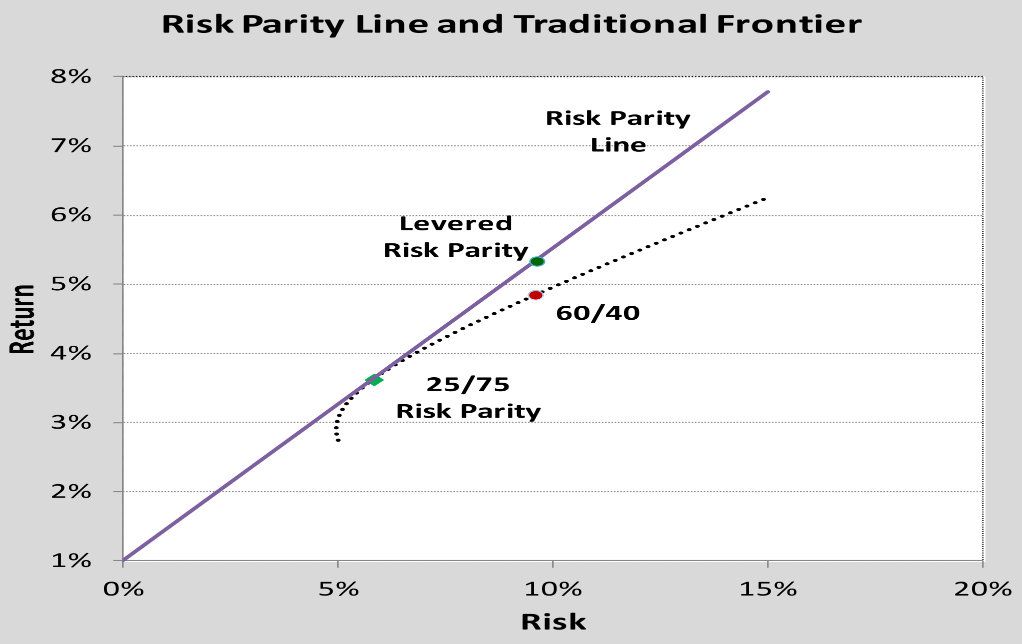
$$\sum_{i=1}^{M} w_i (1 + \mu_i)^N \leq \left(1 + \sum_{i=1}^{M} w_i \mu_i\right)^N$$

$$\mathrm{E}(W_{\mathrm{BH}}) \leq \mathrm{E}(W_{\mathrm{FW}})$$

Long-Short Portfolios

Risk Parity

- The result can be extended to cases with more than one short assets
- Practical application: Risk Parity portfolios
  - Long risky assets: equity, interest rates, commodities, etc.
  - Leveraged by shorting cash



Terminal Wealth

Expected variance

» **Expected value of terminal wealth**

  » **Long-only portfolios:** $\mathrm{E}(W_{\mathrm{BH}}) \geq \mathrm{E}(W_{\mathrm{FW}})$

  » **Long-short portfolios** $\mathrm{E}(W_{\mathrm{BH}}) \leq \mathrm{E}(W_{\mathrm{FW}})$

» **But variance is also important in any investment analysis (risk/return framework)**

  » **What about** $\mathrm{var}(W_{\mathrm{BH}})$ **and** $\mathrm{var}(W_{\mathrm{FW}})$ **?**

Terminal Wealth

Expected variance

» **Statistical calculation**

$$\mathrm{var}(\boldsymbol{x}) = \mathrm{E}(\boldsymbol{x}^2) - [\mathrm{E}(\boldsymbol{x})]^2.$$

» **FW portfolios**

$$\mathrm{var}(W_{\mathrm{FW}}) = \left[\left(1+\mu_p\right)^2 + \sigma_p^2\right]^N - \left(1+\mu_p\right)^{2N}$$

$$\mathrm{var}(W_{\mathrm{FW}}) = \sum_{n=1}^{N} \mathrm{C}_N^n \left(1+\mu_p\right)^{2(N-n)} \sigma_p^{2n}$$

» **BH portfolios**

$$\mathrm{var}(W_{\mathrm{BH}}) = \sum_{i,j=1}^{M} w_i w_j \left[(1+\mu_i)\left(1+\mu_j\right) + \sigma_{ij}\right]^N - \left[\sum_{i=1}^{M} w_i (1+\mu_i)^N\right]^2$$

$$\mathrm{var}(W_{\mathrm{BH}}) = \sum_{n=1}^{N} \mathrm{C}_N^n \sum_{i,j=1}^{M} w_i w_j \left[(1+\mu_i)\left(1+\mu_j\right)\right]^{N-n} \sigma_{ij}^n.$$

Terminal Wealth

Expected variance – special case

» **Theorem: When** $\mu_1 = \mu_2 = \cdots = \mu_M$ **and weights and covariances are non-negative**

» **Then** $\mathrm{var}(W_{\mathrm{BH}}) \geq \mathrm{var}(W_{\mathrm{FW}})$

» **In general, BH long-only portfolios' variance of terminal wealth is higher than that of FW portfolios.**

Risk-adjusted Terminal Wealth

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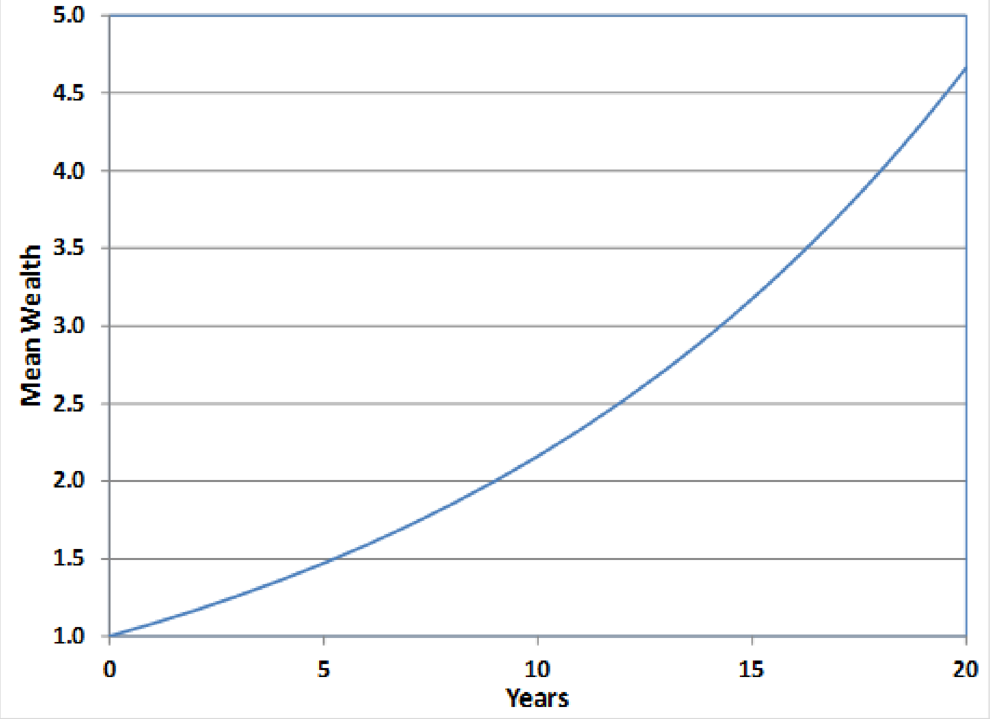
Wealth- volatility ratio

$$\frac{E(W)}{\text{std}(W)}$$

» **Example: 10 securities with equal expected return (8%), equal volatility (20%), equal pair-wise correlation (ρ); initial weight 10% each**



Risk-adjusted Terminal Wealth

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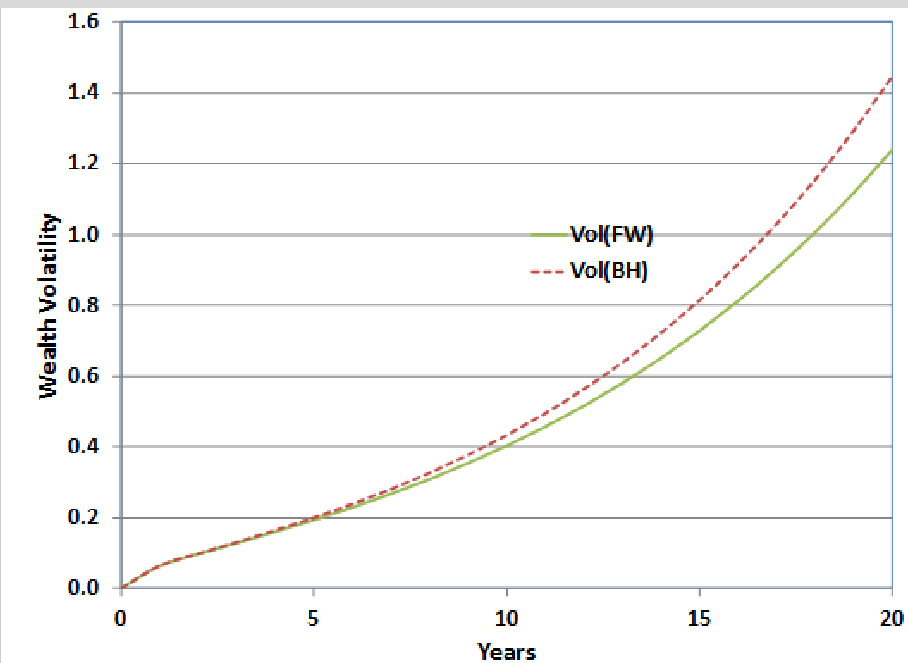
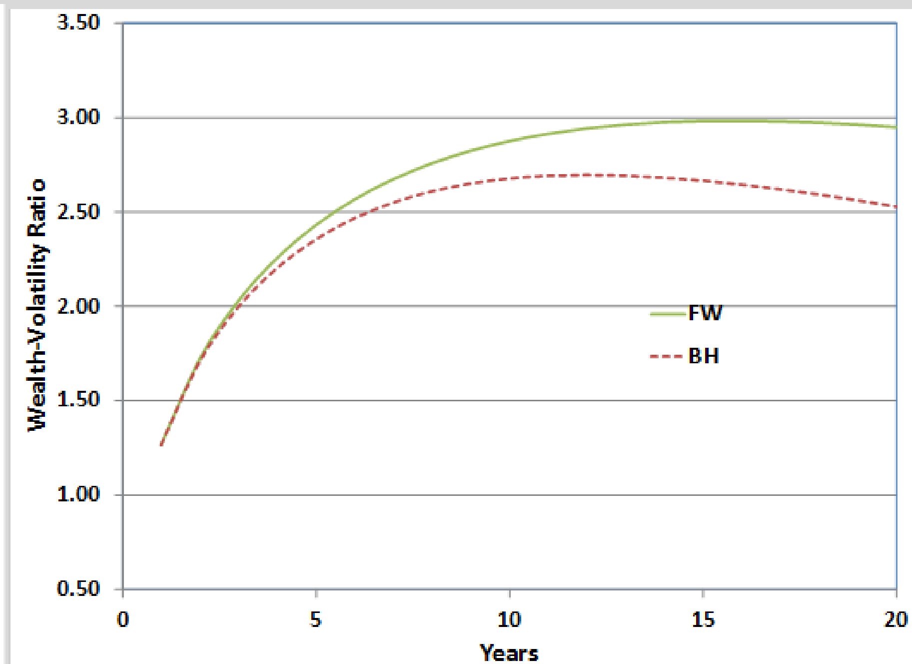
Wealth- volatility ratio

$$\frac{E(W_{\mathrm{FW}})}{\mathrm{std}(W_{\mathrm{FW}})} > \frac{E(W_{\mathrm{BH}})}{\mathrm{std}(W_{\mathrm{FW}})}$$

» **Example: 10 securities with equal expected return (8%), equal volatility (20%), equal pair-wise correlation (ρ=0)**





Risk-adjusted Terminal Wealth

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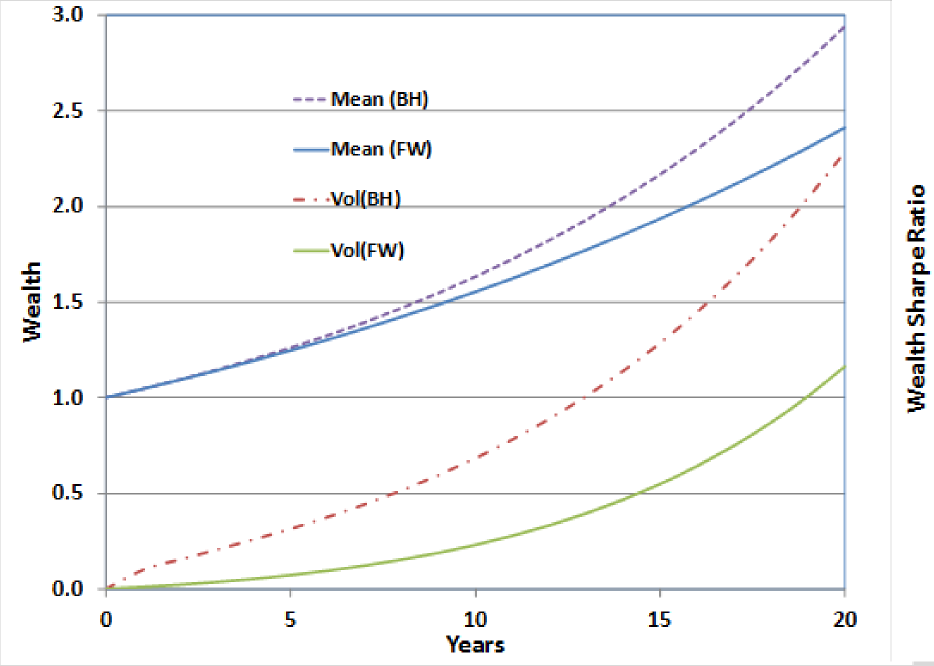
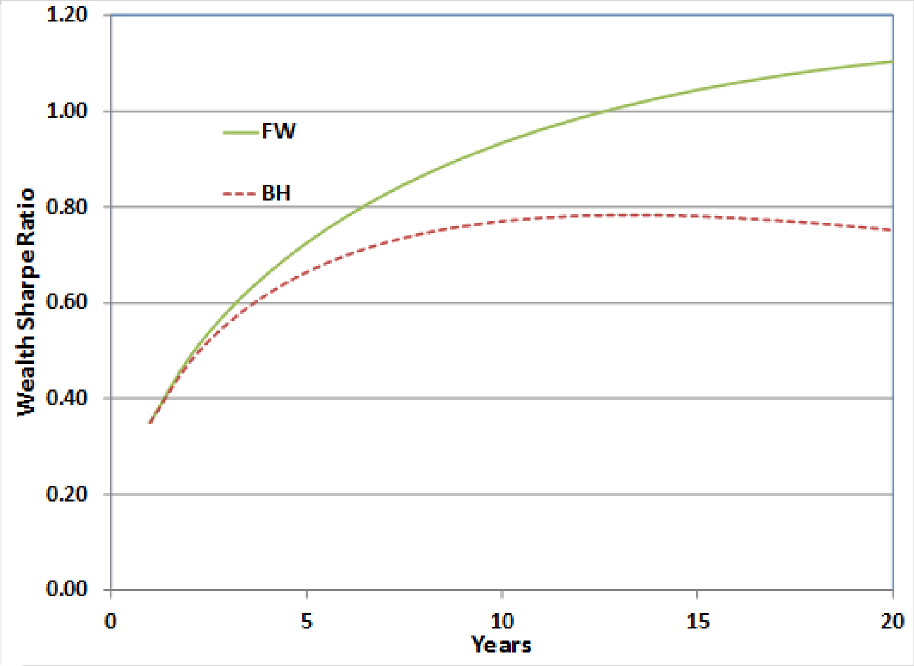
Wealth Sharpe ratio

$$SR_W = \frac{E(W) - (1+\mu_0)^N}{\text{std}(W)}$$

» Example: 2 assets – one risk-free with 1% return and the other 20% risk and 8% return; initial weight 50% each





Effects of Serial Correlations

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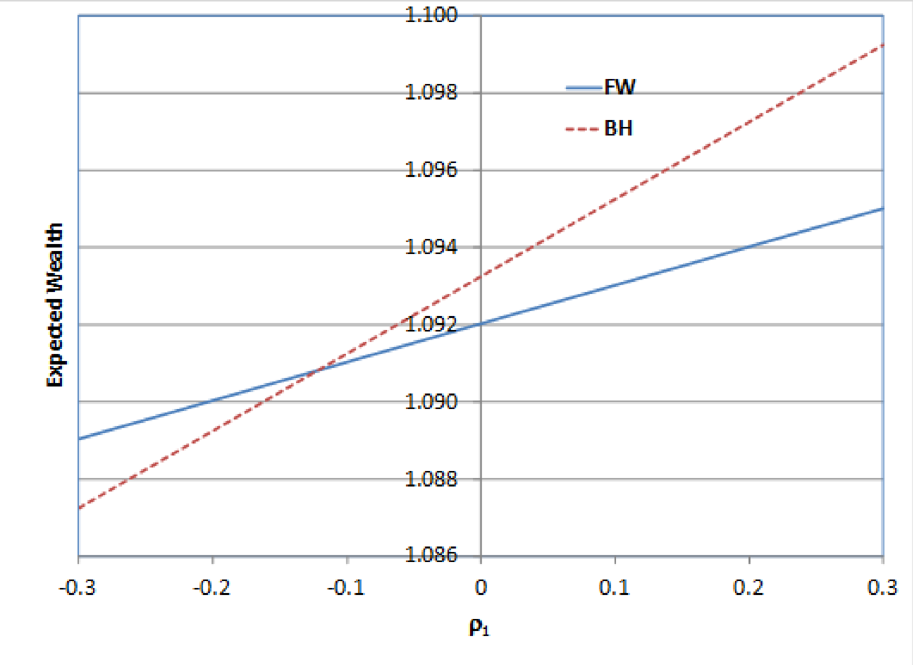
Long- only portfolios

- Mean-reverting gives FW portfolios an edge; trending or momentum gives BH portfolios an edge
- Example: 2 assets – one risk-free with 1% return and the other 20% risk and 8% return; initial weight 50% each



$$\mathrm{E}(W_{\mathrm{BH}}) \leq \mathrm{E}(W_{\mathrm{FW}})$$

$$\text{if } \rho_1 < -\left(\frac{\mu_1 - \mu_0}{\sigma_1}\right)^2.$$

$$\rho_1 < -(SR)^2$$

Conclusions

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To rebalance or not to rebalance

» **Long-only portfolios**

$$E(W_{BH}) \geq E(W_{FW}) \qquad var(W_{BH}) \geq var(W_{FW})$$

» **<u>FW tends to have higher risk-adjusted terminal wealth</u>**

» **Long-short portfolios**

$$E(W_{BH}) \leq E(W_{FW}) \qquad var(W_{BH}) \ ? \ var(W_{FW})$$

» **<u>Buy-and-hold and leveraged portfolio is not a good combination</u>**

» **Serial correlation**

» **Mean-reverting is beneficial to FW long-only portfolios; trending is beneficial to BH long-only portfolios**

» **For long-short portfolios, times series trending and cross-sectional reversal is the best.**

Conclusions

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To rebalance

- Investors often have fixed-weight asset allocation portfolios but buy-and-hold asset indices
- Capitalization-weighted indices are BH and they often underperformed naïve equally-weighted portfolio and other kinds of alternative indices
  - Cap-weighted indices are not diversified; "it is passive-aggressively active."
  - Cap-weighted indices are not rebalanced
- "To rebalance or not to rebalance?" Answer: Rebalance everywhere

Appendix

Diversification "return" is not rebalancing return

» **Arithmetic mean**

$$\mu = \frac{1}{M}\left(r_1 + \cdots + r_M\right)$$

$$g \approx \mu - \frac{1}{2}\sigma^2$$

» **Geometric mean**

$$1 + g = \left[\left(1 + r_1\right)\cdots\left(1 + r_M\right)\right]^{1/M}$$

$$DR = g_p - \sum_{i=1}^{N} w_i g_i \geq 0$$

» **Diversification return is not return between two real portfolios**

» $\sum_{i=1}^{N} w_i g_i$ **IS NOT the geometric mean of the buy-and-hold portfolio**

» **Qian, Edward, "Diversification Return and Leveraged Portfolios",** ***The Journal of Portfolio Management*****, Summer 2012, Vol. 38, No. 4: pp. 14-25**

Appendix

Diversification return

» **Arithmetic mean**

$$\mu = \frac{1}{M}\left(r_1 + \cdots + r_M\right)$$

$$g \approx \mu - \frac{1}{2}\sigma^2$$

» **Geometric mean**

$$1 + g = \left[\left(1 + r_1\right)\cdots\left(1 + r_M\right)\right]^{1/M}$$

$$g_p = \mu_p - \frac{1}{2}\sigma_p^2 = \sum_{i=1}^{N} w_i \mu_i - \frac{1}{2}\sigma_p^2 = \sum_{i=1}^{N} w_i\left(g_i + \frac{1}{2}\sigma_i^2\right) - \frac{1}{2}\sigma_p^2$$

$$g_p - \sum_{i=1}^{N} w_i g_i = \frac{1}{2}\left[\sum_{i=1}^{N} w_i \sigma_i^2 - \sigma_p^2\right] \ge 0$$