Decomposing Variance Risk Measure for Long-Term Investors

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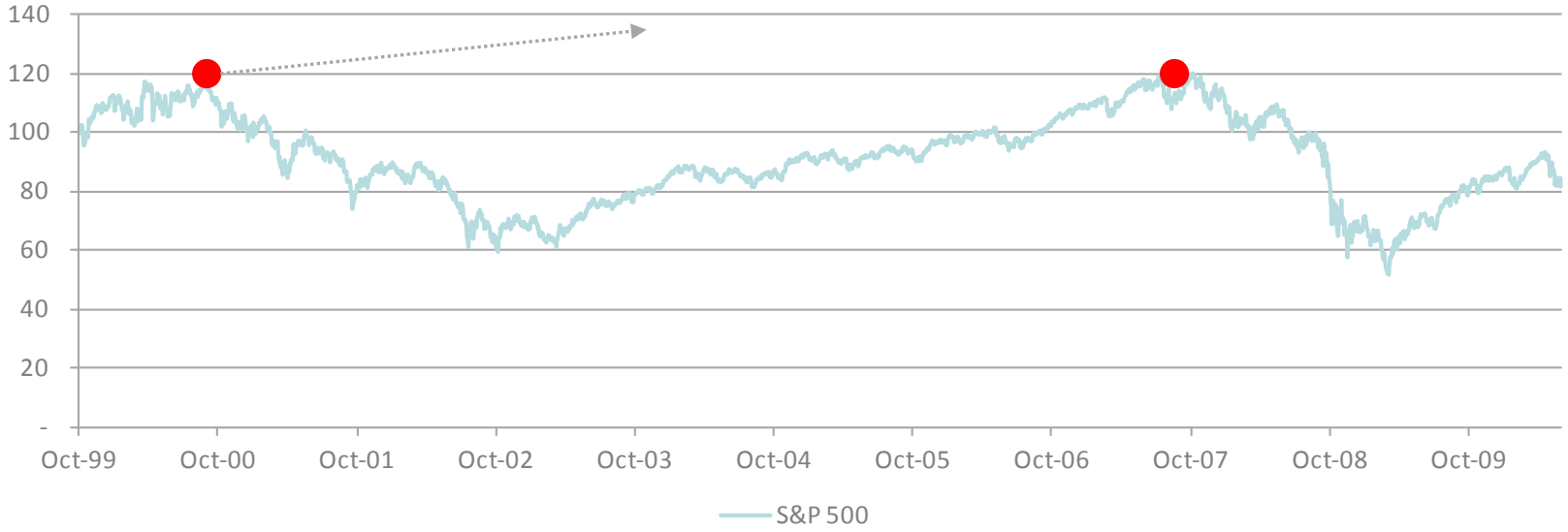
Decomposing Variance Risk for Long-Term Investors

- Risk in Return Space – FD Strategy Overlay
- Risk for Long-Term Investors – BH Strategy

- Long-Term Trend Risk Decomposition
- Long-Term Trend Risk for Long-Term Investors

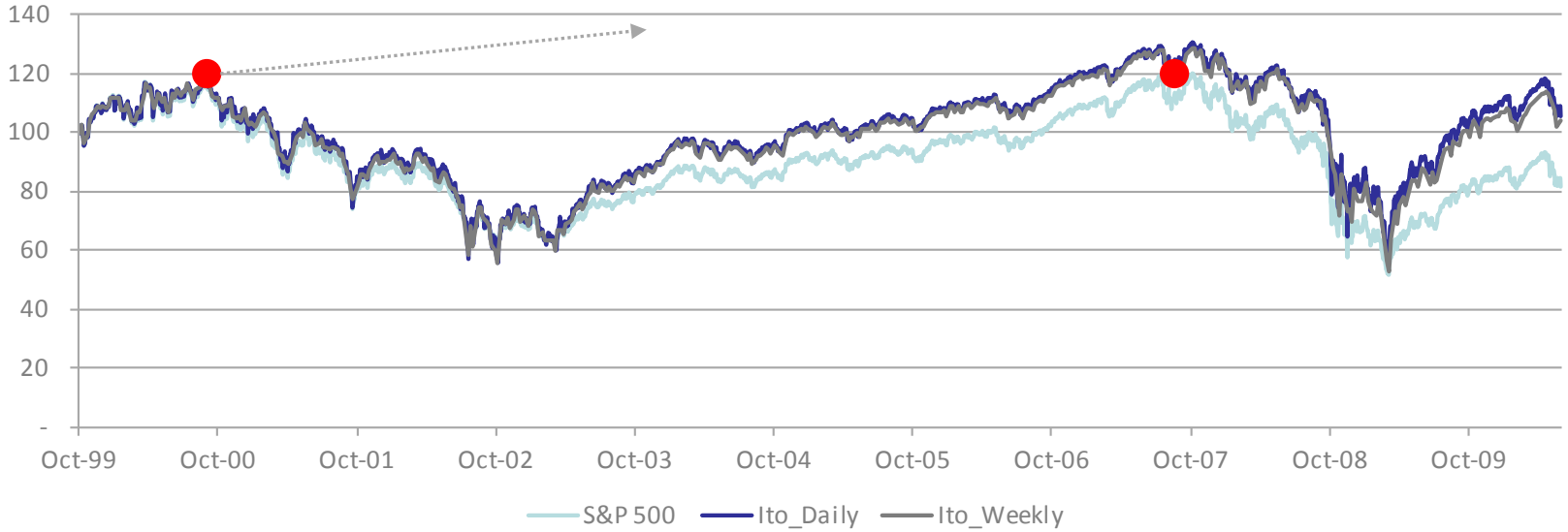
- Systematic Risk, Factor Model etc

Risk in Return Space



$$\frac{\Delta S}{S} - \Delta[\ln(S)] = N\{\mu, \vec{\sigma}\} - N\{\mu - \tfrac{\sigma^2}{2}, \vec{\sigma}\} = \tfrac{\sigma^2}{2}$$

Risk in Return Space

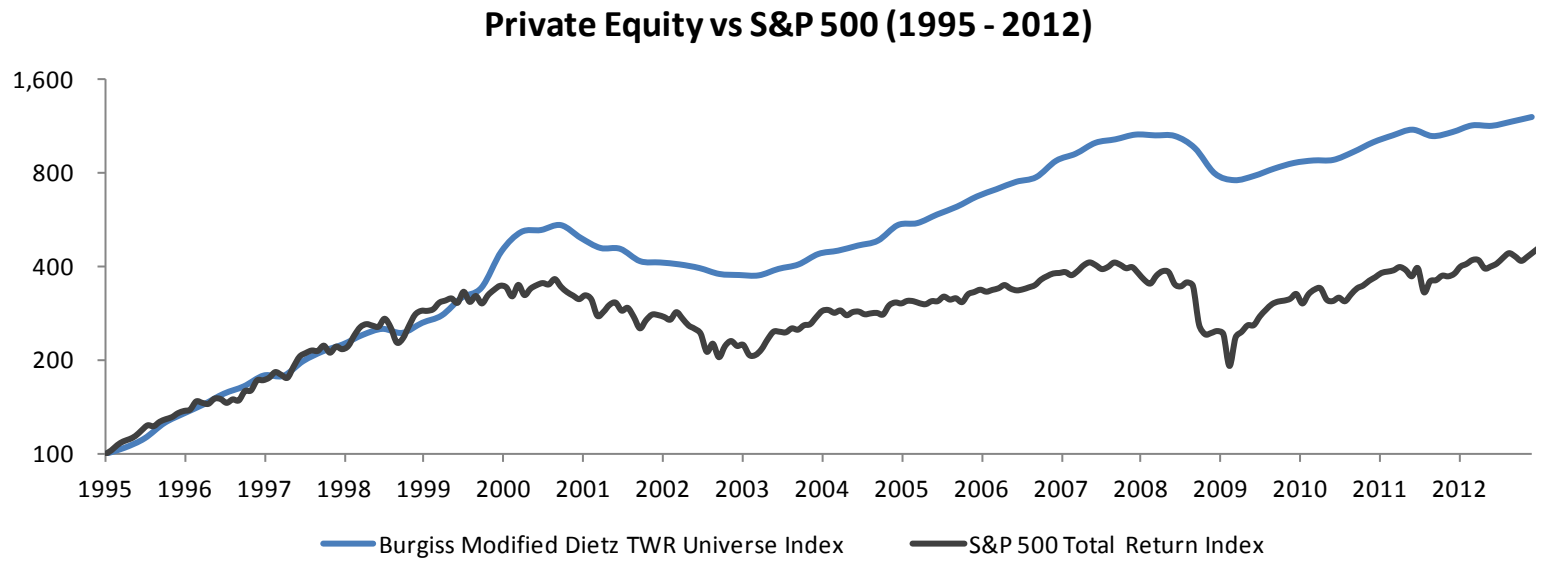


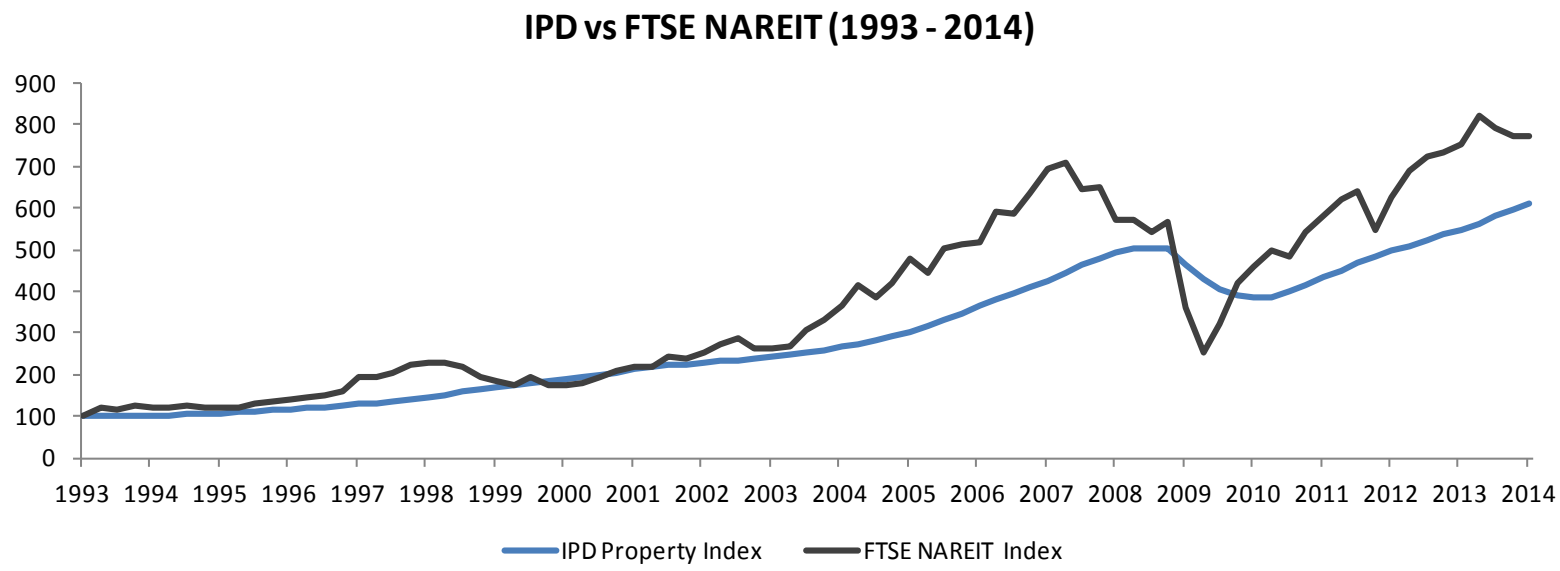
Return Space vs Price/Value Space:

- Strategy overlay
- Distribution model assumption

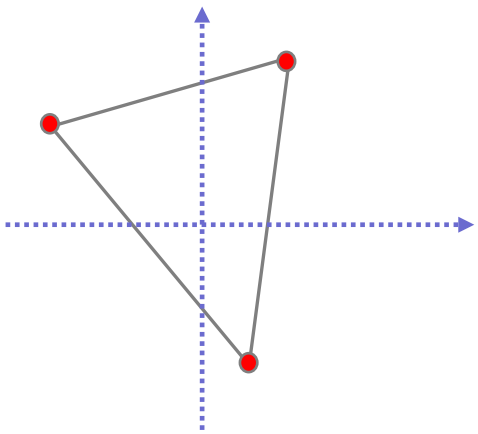
- Return analysis assigns an embedded FD rebalancing strategy.
- Return analysis is about the PnL performance of the FD-strategy, not BH-strategy.
- Active return is more about the active PnL performance of the FD-strategies.

Long-Term Investors: Risk of BH Asset vs FD Strategy



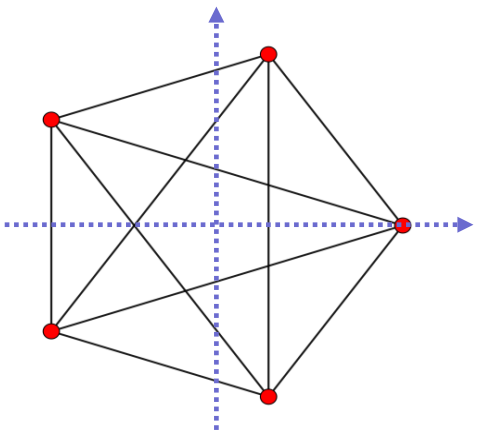


Decomposing the Long-Term Trend Risk



$$\sigma_A^2 = \sum_{i<j} p_i p_j (A_i - A_j)^2$$

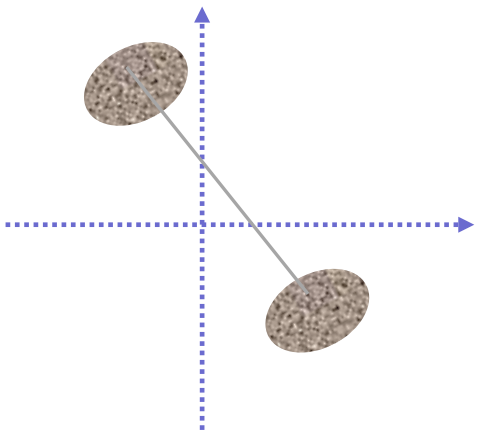
$$\mathrm{cov}(A,B) = \sum_{i<j} p_i p_j (A_i - A_j)(B_i - B_j)$$



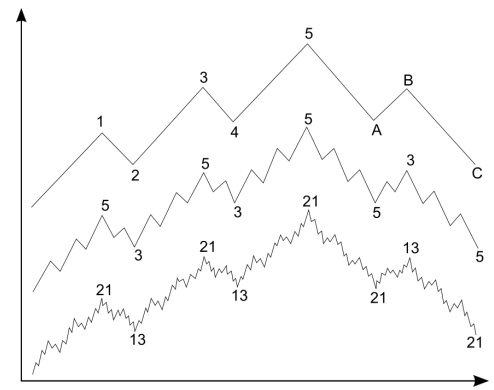
$$\begin{aligned}
\mathrm{cov}(A,B) &\equiv \left\langle (A_i - \overline{A})(B_i - \overline{B}) \right\rangle = \left\langle A_i B_i \right\rangle - \overline{A} \cdot \overline{B} = \sum_i p_i A_i B_i - (\sum_i p_i A_i)(\sum_j p_j B_j) \\
&= \sum_i p_i A_i B_i - \sum_i p_i^2 A_i B_i - \sum_{i \neq j} p_i p_j A_i B_j = \sum_i p_i (1 - p_i) A_i B_i - \sum_{i \neq j} p_i p_j A_i B_j \\
&= \sum_i p_i (\sum_{i \neq j} p_j) A_i B_i - \sum_{i \neq j} p_i p_j A_i B_j = \sum_{i \neq j} p_i p_j A_i B_i - \sum_{i \neq j} p_i p_j A_i B_j \\
&= \sum_{i \neq j} p_i p_j A_i (B_i - B_j) = \sum_{i<j} p_i p_j [A_i (B_i - B_j)] + [A_j (B_j - B_i)]
\end{aligned}$$

$$H = -\sum_i p_i \ln(p_i) \approx -\sum_i p_i (p_i - 1) = \sum_i p_i (\sum_{j \neq i} p_j) = 2 \sum_{i<j} p_i p_j$$

Decomposing the Long-Term Trend Risk



$$\sigma^2_{R_A} = \left\langle \sigma^2_{R_A|m} \right\rangle + \sigma^2_{\langle R_A|m \rangle}$$
$$= \sum_m p_m \sigma^2_{{}_m R_A} + \sum_{m<n} p_m p_n ({}_m\mu_A - {}_n\mu_A)^2$$



Law of Total Variance/Covariance - switch regimes	Short-Term Volatility + Long-Term Trend Risk
• Center of Mass Decomposition - moment of inertia	• Spread Variation + Credit Ratings Migration
• Iterated Decomposition - Elliott wave	• Valuation Uncertainty + Trend Uncertainty
• Variance/Covariance Formula – shuffle returns	• FD Volatility Strategy + FD Charting

$$\mathrm{cov}(R_A, R_B) = \left\langle \mathrm{cov}(R_A|m, R_B|m) \right\rangle + \mathrm{cov}(\langle R_A|m \rangle, \langle R_B|m \rangle)$$
$$= \sum_m p_m \,\mathrm{cov}({}_m R_A, {}_m R_B) + \sum_{m<n} p_m p_n ({}_m\mu_A - {}_n\mu_A)({}_m\mu_B - {}_n\mu_B)$$

$$\left\langle R_A^2 \right\rangle = \left\langle (R_A - \mu_A)^2 \right\rangle + \mu_A^2$$
$$\left\langle R_A R_B \right\rangle = \left\langle (R_A - \mu_A)(R_B - \mu_B) \right\rangle + \mu_A \mu_B$$

$$\left\langle (R_A - a)^2 \right\rangle = \left\langle (R_A - \mu_A)^2 \right\rangle + (\mu_A - a)^2$$
$$\left\langle (R_A - a)(R_B - b) \right\rangle = \left\langle (R_A - \mu_A)(R_B - \mu_B) \right\rangle + (\mu_A - a)(\mu_B - b)$$

Long Risk Horizon with Long-Term Trend Risk

$$\sigma^2_{R_A} = \left\langle \sigma^2_{R_A|m} \right\rangle + \sigma^2_{\langle R_A|m \rangle}$$
$$= \sum_m p_m \sigma^2_{{}_m R_A} + \sum_{m<n} p_m p_n ({}_m\mu_A - {}_n\mu_A)^2$$
$$= \sum_m T_m h_m \sigma^2_{{}_m R_A} + \sum_{m<n} T_m T_n ({}_m\mu_A h_m - {}_n\mu_A h_n)^2$$

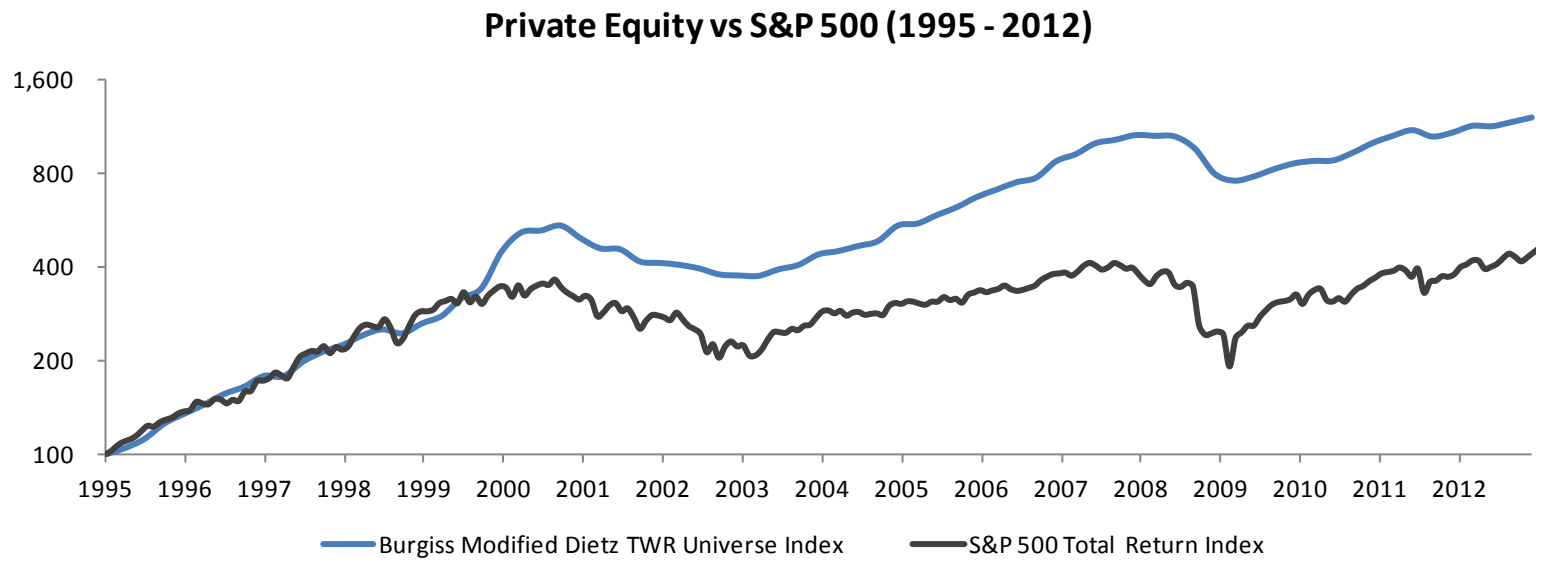
Long-Term Trend Risk

- dependency in risk horizon h^2, where h is capped by min{regime period T }
- for long-term BH investors, it can dominate over short-term volatility, *eg.* in **private market**
- calibrate to direct calculation: $\gtrsim \sqrt{h}$ scaling, then natural "**mean reversion**" from capping
- relates to maximum drawdown, drift uncertainty, auto-correlation, price space risk

$$\sigma^2_{(R_A - R_B)} = \left\langle \sigma^2_{(R_A - R_B)|m} \right\rangle + \sigma^2_{\langle (R_A - R_B)|m \rangle}$$
$$= \sum_m p_m \sigma^2_{({}_m R_A - {}_m R_B)} + \sum_{m<n} p_m p_n [({}_m\mu_A - {}_m\mu_B) - ({}_n\mu_A - {}_n\mu_B)]^2$$
$$= \sum_m p_m [\vec{\sigma}_{{}_m R_A} - \vec{\sigma}_{{}_m R_B}]^2 + \sum_{m<n} p_m p_n [({}_m\mu_A - {}_n\mu_A) - ({}_m\mu_B - {}_n\mu_B)]^2$$

- The formula decomposes contribution to active risk from the **trending gap** between asset and benchmark
- TE(long-term trend) might be more important than TE(short-term volatility), for long-term BH investors

Long Risk Horizon with Long-Term Trend Risk



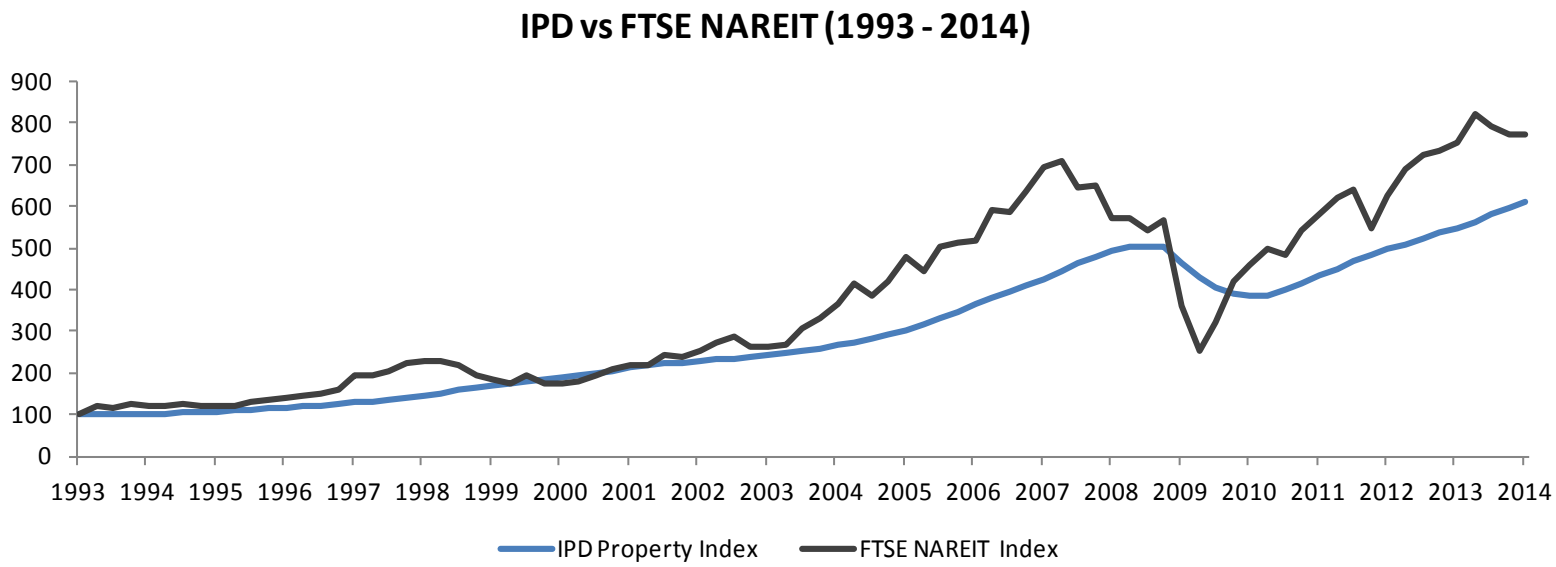
Conventional vs Regime Switching

	Conventional			Regime Switching		
Horizon	Correlation	PE	S&P	Correlation	PE	S&P
1Q	69%	6%	9%	69%	6%	9%
1Y	68%	22%	20%	81%	20%	24%
3Y				82%	47%	46%
5Y				84%	70%	61%
10Y				86%	79%	67%
20Y				86%	79%	67%

Regime Switching Decomposition

	Correlation		PE		S&P	
Horizon	Short	Long	Short	Long	Short	Long
1Q	35%	34%	5%	5%	7%	5%
1Y	17%	64%	9%	18%	15%	18%
3Y	10%	72%	15%	44%	25%	39%
5Y	7%	77%	19%	67%	29%	54%
10Y	6%	80%	20%	76%	30%	60%
20Y	6%	80%	20%	76%	30%	60%

Long Risk Horizon with Long-Term Trend Risk



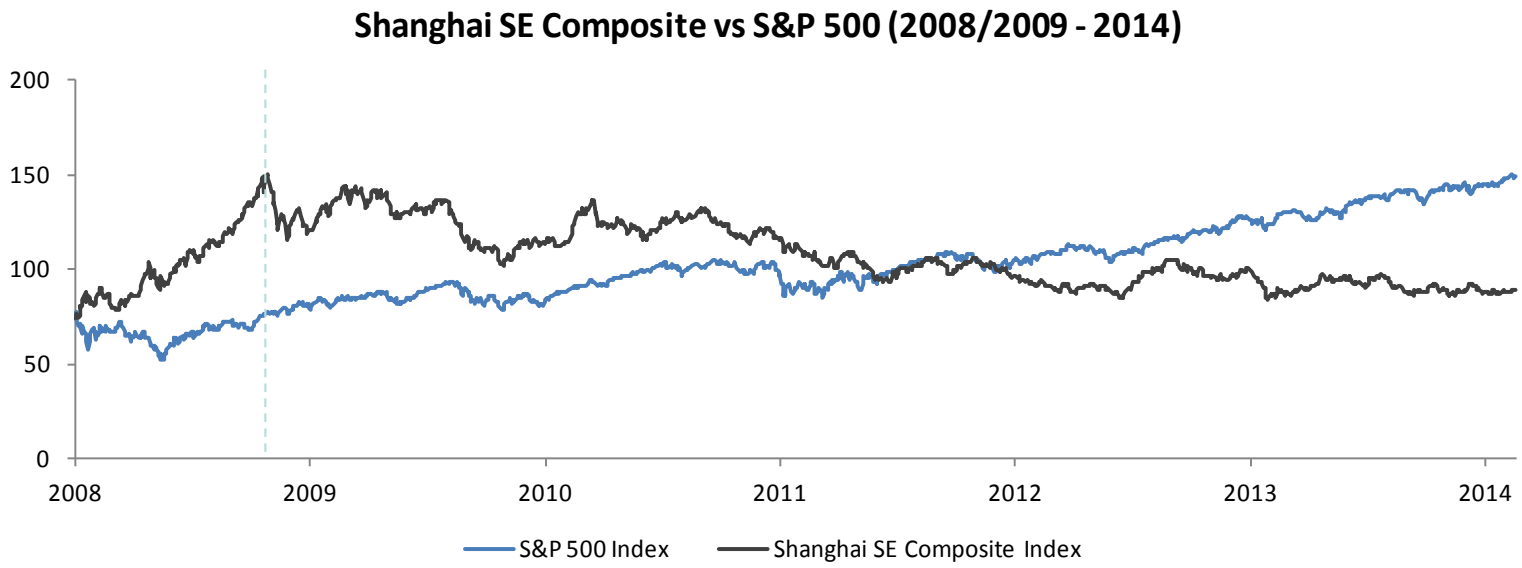
Conventional vs Regime Switching

	Conventional			Regime Switching		
Horizon	Correlation	IPD	NAREIT	Correlation	IPD	NAREIT
1Q	20%	2%	10%	20%	2%	10%
1Y	21%	8%	22%	20%	8%	20%
3Y				31%	16%	31%
5Y				39%	21%	38%
10Y				63%	39%	57%
20Y				77%	63%	83%

Regime Switching Decomposition

	Correlation		IPD		NAREIT	
Horizon	Short	Long	Short	Long	Short	Long
1Q	16%	5%	1%	2%	10%	1%
1Y	9%	11%	3%	7%	20%	2%
3Y	5%	27%	4%	15%	29%	9%
5Y	3%	36%	5%	20%	35%	14%
10Y	1%	61%	7%	39%	45%	36%
20Y	1%	76%	9%	63%	53%	64%

Long Risk Horizon with Long-Term Trend Risk



Conventional vs Regime Switching

	Conventional			Regime Switching		
Horizon	Correlation	China	S&P	Correlation	China	S&P
1D	12%	1%	1%	7%	1%	1%
1W	33%	3%	2%	7%	3%	3%
1M	38%	5%	4%	6%	7%	6%
1Y	57%	9%	8%	-1%	35%	20%
3Y				-19%	47%	31%
5Y				-28%	58%	42%

Regime Switching Decomposition

	Correlation		China		S&P	
Horizon	Short	Long	Short	Long	Short	Long
1D	7%	-0%	1%	0%	1%	0%
1W	7%	-0%	3%	1%	3%	0%
1M	7%	-0%	6%	3%	6%	0%
1Y	5%	-6%	21%	28%	20%	1%
3Y	7%	-26%	33%	34%	29%	11%
5Y	7%	-35%	42%	39%	36%	21%

Long-Term Risk Modeling - Further Points

- **Systematic Risk**: efficient portfolio or uncertain long-term trend?
- **Factor/Regression Model**: how to model long-term trend factor?
- **Diversification**: short-term volatility or long-term risk?
- **Private/Public Asset Allocation**: smoothing or de-smoothing?

Appendix: Bias, Drift Uncertainty, Auto-correlation

- **Variance Definition** $$\left\langle R^2 \right\rangle = \left\langle (R-\overline{R})^2 \right\rangle + \overline{R}^2$$
- **Drift Uncertainty** $$\sigma_p^2 = \left\langle \sigma_s^2 \right\rangle + \sigma_{\overline{R}}^2$$
- **Volatility of Volatility** $$\sigma_p^2 = \left\langle \sigma_s \right\rangle^2 + \sigma_{\sigma_s}^2 + \sigma_{\overline{R}}^2$$

- **Drift Uncertainty (ID):** $$\frac{\sigma_{\overline{R}}^2}{\sigma_p^2} = \frac{1}{n} + \frac{n-1}{n}\overline{\rho} = \overline{\rho} + \frac{1}{n}(1-\overline{\rho})$$
- **Observed Variance:** $$\frac{\left\langle \sigma_s^2 \right\rangle}{\sigma_p^2} = 1 - \frac{\sigma_{\overline{R}}^2}{\sigma_p^2} = \frac{n-1}{n}(1-\overline{\rho})$$
- **Bessel Correction:** $$\hat{\sigma}_s^2 \equiv \frac{n\sigma_s^2}{n-1}\frac{1}{1-\overline{\rho}} = \frac{\sum (R_i - \overline{R})^2}{n-1}\frac{1}{1-\overline{\rho}}$$